OPTIMAL RECOVERY OF FUNCTIONS AND THEIR DERIVATIVES FROM FOURIER COEFFICIENTS GIVEN WITH AN ERROR

G. G. MAGARIL-IL'YAEV, K. YU. OSIPENKO

ABSTRACT. In the paper the problems of optimal recovery of functions and their derivatives from inaccurate values of Fourier coefficients are considered. The explicit expressions of optimal recovery methods for classes of smooth and analytic functions defined on varies compact manifolds are given.

1. STATEMENT OF THE PROBLEM

We begin with the general statement of the optimal recovery problem. Let X be a linear space, Z a normed space, and $T\colon X \to Z$ a linear operator. It is required to recover values of T on the set (class) $W \subset X$ from some information about elements from this class. More precisely, for every element $x \in W$ we have the information $I(x)$ where I is some mapping (which is called *information*) from W to a linear space Y. An information about elements from W may be given inaccurately and therefore I, in general, is a multi-valued mapping.

Every mapping $\varphi\colon Y \to Z$ is admitted as a recovery method. The quantity

$$e(T, W, I, \varphi) = \sup_{\substack{x \in W \\ y \in I(x)}} \|Tx - \varphi(y)\|_Z$$

is called the *error* of such method. The quantity

$$E(T, W, I) = \inf_{\varphi\colon Y \to Z} e(T, W, I, \varphi) \tag{1}$$

is called the *error of optimal recovery* and a method for which the infimum is attained is said to be an *optimal method of recovery* (of the operator T on the class W from the information I).

In the paper we study the situation when X is some subspace of functions from $L_2(M)$ where M is a compact manifold (for example, a circle, d-dimensional sphere, disk in the complex plane), $W \subset X$ is a class of functions such that in particular cases it coincides with several classes of smooth and analytic functions (for example, the Sobolev, Hardy–Sobolev, Bergman–Sobolev classes), $T\colon X \to M$ is a multiplicator type operator which, in particular, is a differential operator, and the information about $x(\cdot) \in W$ is in the fact that we know all or a

This research was carried out with the financial support of the Russian Foundation for Basic Research (grant nos. 99-01-01181 and 00–15–96109).

finite number of Fourier coefficients of the function $x(\cdot)$ with some error (in one or another metric).

When I is a linear operator (that is, the information is given explicitly) the problem of optimal recovery of linear operator in Hilbert spaces was studied in [1]. In the case when the information mapping I with range of values in a Hilbert space is the sum of linear operator and a ball with some radius (defined an error) the corresponding problem was considered in [2] (see also [3]–[5]). In [2] it is proved, in particular, that there exists a linear method among optimal methods of recovery and some algorithm for its finding was proposed. We do not use this result. Our approach is based on standard principles of convex optimization which are a natural tool for solving of such kind problems (see [6]–[8] for solving of recovery problems of linear functionals from general positions of extremum theory). Such approach allows to obtain explicit expressions for optimal methods of recovery also for those cases when the error of information operator is given in the uniform metric.

In this paper, first, the recovery problems for classes of functions defined on a circle are considered in detail. We prove results of rather general type and derive corollaries from them for various classes of smooth and analytic functions. An insignificant modification of these results allows to obtain analogous assertions for classes of functions defined on other manifolds which is illustrated for classes of functions defined on the d-dimensional sphere and unit disk of the complex plane. The list of similar examples may be continued.

We go on to the explicit description of the class W, operator T, and information mappings I in the case when $M=\mathbb{T}$. Let $x(\cdot)$ be from the space $L_2(\mathbb{T})$ with the norm

$$\|x(\cdot)\|_{L_2(\mathbb{T})}=\left(\frac{1}{2\pi}\int_{\mathbb{T}}|x(t)|^2\,dt\right)^{1/2}$$

and

$$x_j=\frac{1}{2\pi}\int_{\mathbb{T}}x(t)e^{-ijt}\,dt,\quad j\in\mathbb{Z},$$

are the Fourier coefficients of $x(\cdot)$. Let $\nu=\{\nu_j\}_{j\in\mathbb{Z}}$ be a sequence of nonnegative numbers. We associate with ν the following subspace in $L_2(\mathbb{T})$

$$X=X^\nu(\mathbb{T})=\Big\{\,x(\cdot)\in L_2(\mathbb{T}):\sum_{j\in\mathbb{Z}}\nu_j|x_j|^2<\infty\,\Big\}$$

and the corresponding class

$$W=W^\nu(\mathbb{T})=\Big\{\,x(\cdot)\in X:\sum_{j\in\mathbb{Z}}\nu_j|x_j|^2\le 1\,\Big\}.$$

Let us give the examples of such type classes. First of all it is the *Sobolev class* $W_2^r(\mathbb{T})$ consisting of 2π-periodic functions $x(\cdot)$ for which the $(r-1)$-st derivative is absolutely continuous and $\|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})} \le 1$. Putting

$$X = \Big\{\, x(\cdot) \in L_2(\mathbb{T}) : \sum_{j\in\mathbb{Z}} j^{2r}|x_j|^2 < \infty \,\Big\}$$

by Plancherel's theorem we get the equivalent definition of the Sobolev class

$$W_2^r(\mathbb{T}) = \Big\{\, x(\cdot) \in X : \sum_{j\in\mathbb{Z}} j^{2r}|x_j|^2 \le 1 \,\Big\}.$$

Thus, $W_2^r(\mathbb{T}) = W^\nu(\mathbb{T})$ where $\nu = \{j^{2r}\}_{j\in\mathbb{Z}}$.

Denote by $\mathcal{H}_2^\beta(\mathbb{T})$ the *Hardy space* of 2π-periodic functions $x(\cdot)$ analytically extended in the strip $S_\beta = \{z \in \mathbb{C} : \operatorname{Im} z| < \beta\}$ and satisfying the condition

$$\|x(\cdot)\|_{\mathcal{H}_2^\beta(\mathbb{T})} = \sup_{0<\rho<\beta} \left(\frac{1}{4\pi}\int_{\mathbb{T}} \left(|x(t+i\rho)|^2 + |x(t-i\rho)|^2\right)\, dt\right)^{1/2} < \infty.$$

The *Bergman space* $\mathcal{A}_2^\beta(\mathbb{T})$ is the set of 2π-periodic functions $x(\cdot)$ analytically extended in the strip S_β and satisfying the condition

$$\|x(\cdot)\|_{\mathcal{A}_2^\beta(\mathbb{T})} = \left(\frac{1}{4\pi\beta}\int_{\mathbb{T}} dt \int_{-\beta}^{\beta} |x(t+i\rho)|^2\, d\rho\right)^{1/2} < \infty.$$

The *Hardy–Sobolev classes* $H_2^{r,\beta}(\mathbb{T})$ and *Bergman–Sobolev classes* $A_2^{r,\beta}(\mathbb{T})$ are defined as the sets of 2π-periodic functions $x(\cdot)$ analytically extended in the strip S_β and satisfying the condition $\|x^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta(\mathbb{T})} \le 1$ and $\|x^{(r)}(\cdot)\|_{\mathcal{A}_2^\beta(\mathbb{T})} \le 1$, respectively.

Functions from the Hardy space $\mathcal{H}_2^\beta(\mathbb{T})$ have boundary values almost everywhere and the space $\mathcal{H}_2^\beta(\mathbb{T})$ is a Hilbert space with the inner product

$$(x(\cdot), y(\cdot))_{\mathcal{H}_2^\beta(\mathbb{T})} = \frac{1}{4\pi}\int_{\mathbb{T}} \Big(x(t+i\beta)\overline{y(t+i\beta)} + x(t-i\beta)\overline{y(t-i\beta)}\Big)\, dt.$$

The Bergman space $\mathcal{A}_2^\beta(\mathbb{T})$ is also a Hilbert space with the inner product

$$(x(\cdot), y(\cdot))_{\mathcal{A}_2^\beta(\mathbb{T})} = \frac{1}{4\pi\beta}\int_{\mathbb{T}} dt \int_{-\beta}^{\beta} x(t+i\rho)\overline{y(t+i\rho)}\, d\rho.$$

The system of functions $\{e^{ij\cdot}\}_{j\in\mathbb{Z}}$ forms an orthogonal basis in the spaces $\mathcal{H}_2^\beta(\mathbb{T})$ and $\mathcal{A}_2^\beta(\mathbb{T})$, moreover

$$\|e^{ij\cdot}\|^2_{\mathcal{W}} = \begin{cases} \cosh 2j\beta, & \mathcal{W} = \mathcal{H}_2^\beta(\mathbb{T}),\\ 1, & \mathcal{W} = \mathcal{A}_2^\beta(\mathbb{T}),\ j = 0,\\ \dfrac{\sinh 2j\beta}{2j\beta}, & \mathcal{W} = \mathcal{A}_2^\beta(\mathbb{T}),\ j \ne 0. \end{cases}$$

Thus, $x(\cdot) \in W = W_2^r(\mathbb{T}), H_2^{r,\beta}(\mathbb{T}), A_2^{r,\beta}(\mathbb{T})$ if and only if

$$x(t) = \sum_{j\in\mathbb{Z}} x_j e^{ijt}$$

and

$$\sum_{j\in\mathbb{Z}} \nu_j(W)|x_j|^2 \le 1,$$

where

$$\nu_j(W) = \begin{cases} j^{2r}, & W = W_2^r(\mathbb{T}), \\ j^{2r}\cosh 2j\beta, & W = H_2^{r,\beta}(\mathbb{T}), \\ j^{2r}\dfrac{\sinh 2j\beta}{2j\beta}, & W = A_2^{r,\beta}(\mathbb{T}). \end{cases}$$

In this case the spaces X are considered as the spaces

$$X = X^{\nu(W)} = \Big\{ x(\cdot) \in L_2(\mathbb{T}) \colon \sum_{j\in\mathbb{Z}} \nu_j(W)|x_j|^2 < \infty \Big\}.$$

The multiplicator type operators $T\colon X \to L_2(\mathbb{T})$ which we study here are defined as follows: if $y(\cdot) = Tx(\cdot)$ and $\{x_j\}_{j\in\mathbb{Z}}$, $\{y_j\}_{j\in\mathbb{Z}}$ are the Fourier coefficients of $x(\cdot)$ and $y(\cdot)$, respectively, then $y_j = \gamma_j x_j$, $j \in \mathbb{Z}$, where $\{\gamma_j\}_{j\in\mathbb{Z}}$ is some sequence of numbers. For example, it is clear that the sequence $\gamma_j = (ij)^k$, $j \in \mathbb{Z}$, corresponds to the differential operator of order $k > 0$.

Finally let us describe information mappings which will be consider.

1. The information $Ix(\cdot) = I_\delta x(\cdot)$ about a function $x(\cdot) \in W$ is in the fact that we have available the numbers $\{y_j\}_{j\in\mathbb{Z}}$ such that

$$\sum_{j\in\mathbb{Z}} |x_j - y_j|^2 \le \delta^2,$$

where $\{x_j\}_{j\in\mathbb{Z}}$ are the Fourier coefficients of $x(\cdot)$ and $\delta > 0$. Formally it means that if

$$Y = l_2 = \Big\{ z = \{z_j\}_{j\in\mathbb{Z}} : \|z\|_{l_2}^2 = \sum_{j\in\mathbb{Z}} |z_j|^2 < \infty \Big\},$$

$F\colon X \to Y$ is a linear operator such that $Fx(\cdot) = \{x_j\}_{j\in\mathbb{Z}}$, and BY is the unit ball of Y, then $I_\delta x(\cdot) = Fx(\cdot) + \delta BY$.

2. The information $Ix(\cdot) = I_\delta^{2N+1}x(\cdot)$ about a function $x(\cdot) \in W$ is in the fact that we have available the numbers $\{y_j\}_{|j|\le N}$ such that

$$\sum_{|j|\le N} |x_j - y_j|^2 \le \delta^2,$$

where $\{x_j\}_{|j|\le N}$ are the first $2N+1$ Fourier coefficients of $x(\cdot)$ and $\delta > 0$. In this case $I_\delta^{2N+1}x(\cdot) = Fx(\cdot) + \delta BY$, where

$$Y = l_2^{2N+1} = \Big\{ z = \{z_j\}_{|j|\le N} : \|z\|_{l_2^{2N+1}}^2 = \sum_{|j|\le N} |z_j|^2 \Big\},$$

$Fx(\cdot) = \{x_j\}_{|j|\le N}$.

3. The information $Ix(\cdot) = I_{\overline{\delta}}^{2N+1}x(\cdot)$ about a function $x(\cdot) \in W$ is in the fact that we have available the numbers $\{y_j\}_{|j|\le N}$ such that $|x_j - y_j| \le \delta_j$, $|j| \le N$, where $\{x_j\}_{|j|\le N}$ are the first $2N+1$ Fourier coefficients of $x(\cdot)$, $\overline{\delta} = \{\delta_j\}_{|j|\le N}$, and $\delta_j > 0$, $|j| \le N$. If

$$Y = l_\infty^{2N+1} = \Big\{ z = \{z_j\}_{|j|\le N} : \|z\|_{l_\infty^{2N+1}} = \sup_{|j|\le N} |z_j| \Big\},$$

$Fx(\cdot) = \{x_j\}_{|j|\le N}$, and

$$B(\overline{\delta}) = \Big\{ z = \{z_j\}_{|j|\le N} : |z_j| \le \delta_j,\ |j| \le N \Big\}, \tag{2}$$

then $I_{\overline{\delta}}^{2N+1}x(\cdot) = Fx(\cdot) + B(\overline{\delta})$.

With some assumptions about the sequence $\nu = \{\nu_j\}_{j\in\mathbb{Z}}$ defined the class W and the sequence $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}}$ defined the operator T we find the error of optimal recovery and optimal recovery method for T on the class W for the all enumerated information mappings. As corollaries we formulate the corresponding assertions for a number of concrete classes of smooth and analytic functions.

2. Statement of main results

Let there be given the sequences $\nu = \{\nu_j\}_{j\in\mathbb{Z}}$ and $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}}$. Set $\mu_j = |\gamma_j|^2$, $j \in \mathbb{Z}$. We shall assume the following conditions are fulfilled:

1) $\{\mu_j\}_{j\in\mathbb{Z}}$ and $\{\nu_j\}_{j\in\mathbb{Z}}$ are even sequences (that is, $\mu_j = \mu_{-j}$ and $\nu_j = \nu_{-j}$, $j \in \mathbb{Z}$) and $\mu_0 = \nu_0 = 0$;
2) $\{\mu_j\}_{j\in\mathbb{N}}$, $\{\nu_j\mu_j^{-1}\}_{j\in\mathbb{N}}$ are positive increasing sequences and $\nu_j \to \infty$ as $j \to \infty$;
3) for all $\lambda_1, \lambda_2 > 0$ the sequence $\{-\mu_j + \lambda_1 + \lambda_2\nu_j\}_{j\in\mathbb{Z}_+}$ has at most two sign changes (with changing of zero terms by arbitrary values ± 1).

2.1. **Recovery from inaccurate information of Fourier coefficients in l_2-metric.** The problem (1) in this case is written as follows

$$E(T, W, I_\delta) = \inf_{\varphi:\, l_2 \to L_2(\mathbb{T})} \sup_{\substack{x(\cdot)\in W,\ y\in l_2 \\ \|Fx(\cdot)-y\|_{l_2}\le\delta}} \|Tx(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{T})}, \tag{3}$$

where $Fx(\cdot) = \{x_j\}_{j\in\mathbb{Z}}$ are the Fourier coefficients of $x(\cdot)$.

Theorem 1. *Let $\{\mu_j\}_{j\in\mathbb{Z}}$ and $\{\nu_j\}_{j\in\mathbb{Z}}$ be sequences satisfying the conditions 1)–3) and T with W are the corresponding operator and class. Then*

$$E(T, W, I_\delta) = \begin{cases} \sqrt{\dfrac{\mu_1}{\nu_1}}, & \delta \ge \nu_1^{-1/2}, \\[2ex] \sqrt{\delta^2\mu_s + (1-\delta^2\nu_s)\dfrac{\mu_{s+1}-\mu_s}{\nu_{s+1}-\nu_s}}, & \nu_{s+1}^{-1/2} \le \delta < \nu_s^{-1/2}, \\ & s \ge 1. \end{cases}$$

Moreover, the method

$$\widehat{\varphi}(y)(\cdot)=\sum_{j\in\mathbb{Z}}\gamma_j\left(1+\nu_j\frac{\mu_{s+1}-\mu_s}{\mu_s\nu_{s+1}-\mu_{s+1}\nu_s}\right)^{-1}y_je^{ij\cdot}$$

is optimal if $\nu_{s+1}^{-1/2}\le\delta<\nu_s^{-1/2}$, $s\ge1$, *and if* $\delta\ge\nu_1^{-1/2}$, *then* $\widehat{\varphi}(y)(\cdot)=0$ *is an optimal method.*

Let us apply Theorem 1 to the optimal recovery problem of the k-th derivative (the corresponding operator we denote by D^k) of a function from the class $W=W_2^r(\mathbb{T}),H_2^{r,\beta}(\mathbb{T}),A_2^{r,\beta}(\mathbb{T})$ by the information I_δ. It is easy to verify that in this case the conditions 1)–3) are fulfilled for $\nu_j=\nu_j(W)$ and $\mu_j=j^{2k}$ (the last of these conditions follows from the fact that for all $\lambda_1,\lambda_2>0$ the sequence $\{(\lambda_1+\lambda_2\nu_j(W))\mu_j^{-1}\}_{j\in\mathbb{N}}$ is convex, that is, its second difference is nonnegative). Thus we have

Corollary 1. *For the optimal recovery error of the k-th derivative of a function from the class* $W=W_2^r(\mathbb{T}),H_2^{r,\beta}(\mathbb{T}),A_2^{r,\beta}(\mathbb{T})$ *by the information* I_δ *the following equality holds:*

$$E(D^k,W,I_\delta)=\nu_1^{-1/2}(W)$$

for $\delta\ge\nu_1^{-1/2}(W)$ *and*

$$E(D^k,W,I_\delta)=\sqrt{\delta^2s^{2k}+(1-\delta^2\nu_s(W))\frac{(s+1)^{2k}-s^{2k}}{\nu_{s+1}(W)-\nu_s(W)}}$$

for $\nu_{s+1}^{-1/2}(W)\le\delta<\nu_s^{-1/2}(W)$, $s\ge1$. *Moreover, the method*

$$\widehat{\varphi}(y)(\cdot)=\sum_{|j|\ge1}(ij)^k\left(1+\nu_j(W)\frac{(s+1)^{2k}-s^{2k}}{s^{2k}\nu_{s+1}(W)-(s+1)^{2k}\nu_s(W)}\right)^{-1}y_je^{ij\cdot}$$

is optimal if $\nu_{s+1}^{-1/2}(W)\le\delta<\nu_s^{-1/2}(W)$, $s\ge1$, *and if* $\delta\ge\nu_1^{-1/2}(W)$, *then* $\widehat{\varphi}(y)(\cdot)=0$ *is an optimal method.*

For recovery of functions themselves ($k=0$) from classes $W=W_2^r(\mathbb{T}),H_2^{r,\beta}(\mathbb{T}),A_2^{r,\beta}(\mathbb{T})$ the following result holds:

$$E(\mathrm{Id},W,I_\delta)=\delta, \tag{4}$$

where Id is the identity operator, and

$$\widehat{\varphi}(y)(\cdot)=\sum_{j\in\mathbb{Z}}y_je^{ij\cdot} \tag{5}$$

is an optimal method.

2.2. **Recovery from inaccurate information of Fourier coefficients in l_2^{2N+1}-metric.** In this case the problem (1) has the form:

$$E(T,W,I_\delta^{2N+1})=\inf_{\varphi\colon l_2^{2N+1}\to L_2(\mathbb{T})}\ \sup_{\substack{x(\cdot)\in W,\ y\in l_2^{2N+1}\\ \|Fx(\cdot)-y\|_{l_2^{2N+1}}\le\delta}}\|Tx(\cdot)-\varphi(y)(\cdot)\|_{L_2(\mathbb{T})},$$

where $Fx(\cdot)=\{x_j\}_{|j|\le N}$ are the first $2N+1$ Fourier coefficients of $x(\cdot)$.

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. Set*

$$s_0=s_0(N)=\min\left\{s\in\mathbb{N}:\frac{\mu_{s+1}-\mu_s}{\nu_{s+1}-\nu_s}\le\frac{\mu_{N+1}}{\nu_{N+1}}\right\}. \tag{6}$$

Then for $\delta\ge\nu_{s_0}^{-1/2}$

$$E(T,W,I_\delta^{2N+1})=E(T,W,I_\delta)$$

and the method

$$\widehat{\varphi}(y)(\cdot)=\sum_{|j|\le N}\gamma_j\left(1+\nu_j\frac{\mu_{s+1}-\mu_s}{\mu_s\nu_{s+1}-\mu_{s+1}\nu_s}\right)^{-1}y_je^{ij\cdot}$$

is optimal if $\nu_{s+1}^{-1/2}\le\delta<\nu_s^{-1/2}$, $1\le s\le s_0-1$, *and if* $\delta\ge\nu_1^{-1/2}$, *then* $\widehat{\varphi}(y)(\cdot)=0$ *is an optimal method. For* $0<\delta<\nu_{s_0}^{-1/2}$

$$E(T,W,I_\delta^{2N+1})=\sqrt{\delta^2\mu_{s_0}+(1-\delta^2\nu_{s_0})\frac{\mu_{N+1}}{\nu_{N+1}}}$$

and

$$\widehat{\varphi}(y)(\cdot)=\sum_{|j|\le N}\gamma_j\left(1+\nu_j\frac{\mu_{N+1}}{\mu_{s_0}\nu_{N+1}-\mu_{N+1}\nu_{s_0}}\right)^{-1}y_je^{ij\cdot}$$

is an optimal method.

It is easy to see that for $M\ge N$

$$E(T,W,I_\delta^{2N+1})\ge E(T,W,I_\delta^{2M+1})\ge E(T,W,I_\delta).$$

Therefore it follows from Theorem 2 that for $\delta\ge\delta_N=\nu_{s_0}^{-1/2}$ the extension of number of Fourier coefficients knowing with the same error δ does not lead to decrease of optimal recovery error.

Thus for a fixed level of error δ the minimal number of the first Fourier coefficients (without taking into account the zero coefficient since in view of the condition $\mu_0=0$ it is not used in the optimal method) which we have to know for maximal precise recovery of the operator T equals $2N_0$ where

$$N_0=\min\{\,N\in\mathbb{N}:\delta_N\le\delta\,\}.$$

By the obvious way the analogue of Corollary 1 for the problem of recovery of the k-th derivative of a function from the class $W=$

$W_2^r(\mathbb{T})$, $H_2^{r,\beta}(\mathbb{T})$, $A_2^{r,\beta}(\mathbb{T})$ by the information I_δ^{2N+1} may be formulated. As to recovery of functions themselves the following result is valid:

$$E(\mathrm{Id}, W, I_\delta^{2N+1}) = \sqrt{\delta^2 + \nu_{N+1}^{-1}(W)}$$

and

$$\widehat{\varphi}(y)(\cdot) = \sum_{|j|\le N} \left(1 + \frac{\nu_j(W)}{\nu_{N+1}(W)}\right)^{-1} y_j e^{ij\cdot}$$

is an optimal method.

2.3. **Recovery from inaccurate information of Fourier coefficients in the uniform metric.** In this case the problem (1) is written as follows:

$$E(T, W, I_{\overline{\delta}}^{2N+1}) = \inf_{\varphi\colon l_\infty^{2N+1}\to L_2(\mathbb{T})} \sup_{\substack{x(\cdot)\in W,\ y\in l_\infty^{2N+1}\\ Fx(\cdot)-y\in B(\overline{\delta})}} \|Tx(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{T})},$$

where $Fx(\cdot) = \{x_j\}_{|j|\le N}$ are the first $2N+1$ Fourier coefficients of $x(\cdot)$ and $B(\overline{\delta})$ is the parallelepiped defined by (2).

Theorem 3. *Let $\mu_j, \nu_j > 0$, $j \in \mathbb{N}$, $\{\nu_j\mu_j^{-1}\}_{j\in\mathbb{N}}$ be an increasing sequence, $\nu_0 = 0$, $\{\mu_j\}_{j\in\mathbb{Z}}$ and $\{\nu_j\}_{j\in\mathbb{Z}}$ be even sequences, and T with W be the corresponding operator and class. Set*

$$p_0 = p_0(\overline{\delta}) = \max\Big\{\, p \in \mathbb{Z}_+ : \sum_{|j|\le p} \nu_j\delta_j^2 < 1,\ \ 0 \le p \le N \,\Big\}.$$

Then

$$E(T, W, I_{\overline{\delta}}^{2N+1}) = \sqrt{\frac{\mu_{p_0+1}}{\nu_{p_0+1}} + \sum_{|j|\le p_0} \left(\mu_j - \nu_j \frac{\mu_{p_0+1}}{\nu_{p_0+1}}\right)\delta_j^2},$$

moreover, the method

$$\widehat{\varphi}(y)(\cdot) = \gamma_0 y_0 + \sum_{1\le|j|\le p_0} \gamma_j \left(1 - \frac{\mu_{p_0+1}\nu_j}{\nu_{p_0+1}\mu_j}\right) y_j e^{ij\cdot}$$

is optimal.

This theorem is also applied to the problem of optimal recovery of functions and their derivatives from the classes $W = W_2^r(\mathbb{T})$, $H_2^{r,\beta}(\mathbb{T})$, $A_2^{r,\beta}(\mathbb{T})$ by the information $I_{\overline{\delta}}^{2N+1}$. As previously in this case for the statement of the corresponding result one must put in Theorem 3 $\mu_j = j^{2k}$, $\nu_j = \nu_j(W)$, $j \in \mathbb{Z}$.

3. Proofs

Before the direct proof of the formulated above theorems we note two assertions of general nature and then describe the scheme by which these theorems will be proved.

Lemma 1. *Let in the problem* (1)

$$\operatorname{gr} I = \{\, (x, y) \in X \times Y : x \in W,\ y \in I(x) \,\}$$

be a convex centrally symmetric set. Then

$$E(T, W, I) \geq \sup_{\substack{x \in W \\ x \in I^{-1}(0)}} \|Tx\|_Z,$$

where $I^{-1}(0) = \{x \in W : 0 \in I(x)\}$.

Proof. For any method φ for all $x \in W$ such that $x \in I^{-1}(0)$ we have

$$2\|Tx\|_Z \leq \|Tx - \varphi(0)\|_Z + \|T(-x) - \varphi(0)\|_Z \leq 2e(T, W, I, \varphi).$$

Consequently, for any method φ

$$e(T, W, I, \varphi) \geq \sup_{\substack{x \in W \\ x \in I^{-1}(0)}} \|Tx\|_Z$$

from which the estimate being proved immediately follows. □

Lemma 2. *Let X, Y, Z, W, T, and I be the same as in the problem* (1)*, Y_0 be a linear space with the semi-inner product $(\cdot, \cdot)_{Y_0}$ and corresponding semi-norm $\|\cdot\|_{Y_0}$, $I_0 \colon X \to Y_0$, $S \colon Y \to Y_0$ be linear operators, and*

$$\operatorname{gr} I \subset \{\, (x, y) \in X \times Y : \|I_0 x - Sy\|_{Y_0} \leq 1 \,\}.$$

Let $\psi \colon Y \to X$ be the mapping which associates with $y \in Y$ a solution of the extremal problem

$$\|I_0 x - Sy\|^2_{Y_0} \to \min, \quad x \in X. \tag{7}$$

Then for the error of the method $\widehat{\varphi} = T \circ \psi$ the following estimate

$$e(T, W, I, \widehat{\varphi}) \leq \sup_{\|I_0 x\|_{Y_0} \leq 1} \|Tx\|_Z$$

holds

Proof. It is easy to verify that in order that $\widehat{x} \in X$ be a solution of (7) it is necessary and sufficient that the relation

$$(I_0 \widehat{x} - Sy, I_0 x)_{Y_0} = 0 \quad \forall x \in X \tag{8}$$

holds. Let $(x, y) \in \operatorname{gr} I$ (that is, $x \in W$ and $y \in I(x)$), then by the assumption $\|I_0 x - Sy\|_{Y_0} \leq 1$. Since $\psi(y)$ is a solution of (7), (8) is fulfilled (changing $\widehat{x}$ by $\psi(y)$) and therefore

$$\|I_0 x - Sy\|^2_{Y_0} = \|I_0 x - I_0(\psi(y))\|^2_{Y_0} + \|I_0(\psi(y)) - Sy\|^2_{Y_0}.$$

Consequently,

$$\|I_0(x-\psi(y))\|_{Y_0} \le \|I_0x - Sy\|_{Y_0} \le 1$$

and hence

$$\|Tx-\widehat{\varphi}(y))\|_Z = \|Tx-T(\psi(y))\|_Z = \|T(x-\psi(y))\|_Z \le \sup_{\|I_0h\|_{Y_0}\le 1} \|Th\|_Z.$$

□

The further scheme of the proof of theorems is the following. The lower bound (Lemma 1) and upper bound (Lemma 2) for the error of optimal recovery are values of extremal problems. In view of definitions of the class W and operator T these problems are reduced to convex programming problems. Using standard methods of convex optimization we find the value of the problem corresponding to the lower estimate and show that it coincides with the value of the problem corresponding to the upper bound for some Y_0, I_0, and S. In view of Lemmas 1 and 2 it means that this value is the error of optimal recovery and the method from Lemma 2 is an optimal method of recovery (in addition it appears that it is linear).

Proof of Theorem 1. 1. The lower bound. According to Lemma 1 and (3) we have to find a solution of the following problem

$$\|Tx(\cdot)\|_{L_2(\mathbb{T})} \to \max, \quad \|Fx(\cdot)\|_{l_2} \le \delta, \ x(\cdot) \in W. \tag{9}$$

Going over to Fourier transforms by virtue of Parseval equality and definitions of operator T and class W we obtain that this problem (changing $\|Tx(\cdot)\|_{L_2(\mathbb{T})}$ by $\|Tx(\cdot)\|^2_{L_2(\mathbb{T})}$) may be written in the following way

$$\sum_{j\in\mathbb{Z}} \mu_j|x_j|^2 \to \max, \quad \sum_{j\in\mathbb{Z}} |x_j|^2 \le \delta^2, \quad \sum_{j\in\mathbb{Z}} \nu_j|x_j|^2 \le 1. \tag{10}$$

Setting $u_j = |x_j|^2$, $j \in \mathbb{Z}$, we rewrite the problem (10) in the form

$$\sum_{j\in\mathbb{Z}} \mu_j u_j \to \max, \quad \sum_{j\in\mathbb{Z}} u_j \le \delta^2, \quad \sum_{j\in\mathbb{Z}} \nu_j u_j \le 1, \quad u_j \ge 0. \tag{11}$$

It is a problem of convex (even linear) programming. Associate with it the Lagrange function ($u = \{u_j\}_{j\in\mathbb{Z}}$)

$$\mathcal{L} = \mathcal{L}(u,\lambda_0,\lambda_1,\lambda_2) = \sum_{j\in\mathbb{Z}} (\lambda_0\mu_j + \lambda_1 + \lambda_2\nu_j)u_j,$$

where $\lambda_0 \le 0$, $\lambda_1, \lambda_2 \ge 0$ are Lagrange multipliers. According to Kuhn-Tucker's theorem if $\widehat{u} = \{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is a solution of the problem (11), then there exist such Lagrange multipliers $\widehat{\lambda}_0 \le 0$, $\widehat{\lambda}_1, \widehat{\lambda}_2 \ge 0$, not all equal zero that the conditions

$$(a) \quad \min_{u_j\ge 0}\mathcal{L}(u,\widehat{\lambda}_0,\widehat{\lambda}_1,\widehat{\lambda}_2)=\mathcal{L}(\widehat{u},\widehat{\lambda}_0,\widehat{\lambda}_1,\widehat{\lambda}_2),$$

$$(b) \quad \widehat{\lambda}_1\Big(\sum_{j\in\mathbb{Z}}\widehat{u}_j-\delta^2\Big)=0,\quad \widehat{\lambda}_2\Big(\sum_{j\in\mathbb{Z}}\nu_j\widehat{u}_j-1\Big)=0$$

hold. If for an admissible sequence in (11) $\widehat{u}=\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ the conditions (a) and (b) are fulfilled with $\widehat{\lambda}_0<0$, then $\widehat{u}$ is a solution of the problem (11).

The last assertion may be easily verified. Indeed, let $u=\{u_j\}_{j\in\mathbb{Z}}$ is an admissible sequence in (11). Then taking into account (a) and (b) we have

$$\widehat{\lambda}_0\sum_{j\in\mathbb{Z}}\mu_ju_j\ge\widehat{\lambda}_0\sum_{j\in\mathbb{Z}}\mu_ju_j+\widehat{\lambda}_1\Big(\sum_{j\in\mathbb{Z}}u_j-\delta^2\Big)+\widehat{\lambda}_2\Big(\sum_{j\in\mathbb{Z}}\nu_ju_j-1\Big)$$

$$\overset{(a)}{\ge}\widehat{\lambda}_0\sum_{j\in\mathbb{Z}}\mu_j\widehat{u}_j+\widehat{\lambda}_1\Big(\sum_{j\in\mathbb{Z}}\widehat{u}_j-\delta^2\Big)+\widehat{\lambda}_2\Big(\sum_{j\in\mathbb{Z}}\nu_j\widehat{u}_j-1\Big)\overset{(b)}{=}\widehat{\lambda}_0\sum_{j\in\mathbb{Z}}\mu_j\widehat{u}_j,$$

that is, $\widehat{u}=\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is a solution of the problem (11).

Now we present such $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ and an admissible sequence $\widehat{u}=\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ for which the conditions (a) and (b) are fulfilled with $\widehat{\lambda}_0=-1$. Then by proved above $\widehat{u}$ is a solution of the problem (11). The form of $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, and $\widehat{u}$ follows from the analysis of relations (a) and (b). Indeed, the sequence $f_j=-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j$, $j\in\mathbb{Z}$, have to be nonnegative and a solution have to be concentrated at the zeros of $\{f_j\}_{j\in\mathbb{Z}}$. In view of the condition on sign changes of this sequence (the condition 3)) positive zeros of $\{f_j\}_{j\in\mathbb{Z}}$ may be followed only in succession. These observations make possible to present corresponding $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, and $\widehat{u}$.

First let $0<\delta<\nu_1^{-1/2}$. Since $\nu_j\to\infty$ as $j\to\infty$ there exists $s\ge 1$ for which $\nu_{s+1}^{-1/2}\le\delta<\nu_s^{-1/2}$. Let us find $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from the condition $f_s=f_{s+1}=0$. Then we obtain

$$\widehat{\lambda}_1=\frac{\mu_s\nu_{s+1}-\mu_{s+1}\nu_s}{\nu_{s+1}-\nu_s},\quad \widehat{\lambda}_2=\frac{\mu_{s+1}-\mu_s}{\nu_{s+1}-\nu_s}. \tag{12}$$

From assumptions about sequences $\{\mu_j\}_{j\in\mathbb{Z}}$ and $\{\nu_j\}_{j\in\mathbb{Z}}$ it follows that $\widehat{\lambda}_1,\widehat{\lambda}_2>0$. With obtained $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ the sequence $\{f_j\}_{j\in\mathbb{Z}}$ is nonnegative by virtue of the condition on sign changes of this sequence. Now we find the sequence $\widehat{u}$ from the condition that it concentrates at the points s and $s+1$ and the conditions (b) are fulfilled, that is, $u_s+u_{s+1}=\delta^2$ and $\nu_su_s+\nu_{s+1}u_{s+1}=1$. Hence we obtain that

$$\widehat{u}_s=\frac{\delta^2\nu_{s+1}-1}{\nu_{s+1}-\nu_s},\quad \widehat{u}_{s+1}=\frac{1-\delta^2\nu_s}{\nu_{s+1}-\nu_s} \tag{13}$$

($\widehat{u}_s\ge 0$, $\widehat{u}_{s+1}>0$ by virtue of the condition on δ). Set $\widehat{u}_j=0$ for $j\ne s,s+1$. Thus $\widehat{u}$ is an admissible sequence. The conditions (a) and

(b) are fulfilled with the obtained $\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{u}$ for $\widehat{\lambda}_0 = -1$ and therefore $\widehat{u}$ is a solution of the problem (11).

Now let $\delta \ge \nu_1^{-1/2}$. Set $\widehat{\lambda}_1 = 0$, $\widehat{\lambda}_2 = \mu_1\nu_1^{-1}$. Then $f_j = 0$ for $|j| \le 1$ and $f_j \ge 0$ for $|j| \ge 2$. Set $\widehat{u}_1 = \nu_1^{-1}$ and $\widehat{u}_j = 0$ for $j \ne 1$. Since $\widehat{u}_1 = \nu_1 \le \delta^2$, $\widehat{u}$ is an admissible sequence. The conditions (a) and (b) are obviously fulfilled with $\widehat{\lambda}_0 = -1$ and consequently, $\widehat{u}$ is a solution of the problem (11).

Thus a solution of the problem (11) is found for all $\delta > 0$, and that means that we found a solution of the problems (10) and (9). Substituting $\widehat{u}$ in the maximizing functional in (11) and extracting square root, we obtain the value of the problem (9) which gives the lower bound for the error of optimal recovery

$$E(T, W, I_\delta) \ge \begin{cases} \sqrt{\dfrac{\mu_1}{\nu_1}}, & \delta \ge \nu_1^{-1/2}, \\ \sqrt{\delta^2\mu_s + (1-\delta^2\nu_s)\dfrac{\mu_{s+1}-\mu_s}{\nu_{s+1}-\nu_s}}, & \nu_{s+1}^{-1/2} \le \delta < \nu_s^{-1/2}, \\ & s \ge 1. \end{cases}$$

2. The upper bound. For the obtained above $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ put $\widehat{\lambda} = \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2$ and $\widehat{\alpha} = \widehat{\lambda}_2(\widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2)^{-1}$. For the obtained solution $\widehat{u}$ of (11) the condition (a) may be rewritten in the form (with $\widehat{\lambda}_0 = -1$)

$$(a_1) \quad \min_{u_j \ge 0} \sum_{j\in\mathbb{Z}} (-\mu_j + \widehat{\lambda}((1-\widehat{\alpha})\delta^{-2} + \widehat{\alpha}\nu_j))u_j = \sum_{j\in\mathbb{Z}} (-\mu_j + \widehat{\lambda}((1-\widehat{\alpha})\delta^{-2} + \widehat{\alpha}\nu_j))\widehat{u}_j.$$

Furthermore, it is easy to verify that the relation

$$(b_1) \quad (1-\widehat{\alpha})\delta^{-2}\sum_{j\in\mathbb{Z}} \widehat{u}_j + \widehat{\alpha}\sum_{j\in\mathbb{Z}} \nu_j\widehat{u}_j = 1$$

holds. By the same arguments as above these conditions are sufficient for $\widehat{u}$ to be a solution of the problem

$$(14) \quad \sum_{j\in\mathbb{Z}} \mu_j u_j \to \max, \quad (1-\widehat{\alpha})\delta^{-2}\sum_{j\in\mathbb{Z}} u_j + \widehat{\alpha}\sum_{j\in\mathbb{Z}} \nu_j u_j \le 1, \quad u_j \ge 0.$$

Thus the values of the problems (11) and (14) coincide.

We show that the problem from Lemma 2 may be reduced to the problem (14). Indeed, let $Y_0 = l_2 \times l^\nu$ where

$$l^\nu = \left\{ z = \{z_j\}_{j\in\mathbb{Z}} : \sum_{j\in\mathbb{Z}} \nu_j |z_j|^2 < \infty \right\}.$$

Define the semi-inner product on Y_0

$$((x,z),(x',z'))_{Y_0} = (1-\widehat{\alpha})\delta^{-2}\sum_{j\in\mathbb{Z}} x_j\overline{x}'_j + \widehat{\alpha}\sum_{j\in\mathbb{Z}}\nu_j z_j\overline{z}'_j.$$

Let the operator $I_0\colon X \to Y_0$ be defined by the equality $I_0x(\cdot) = (Fx(\cdot), Fx(\cdot))$ and $S\colon l_2 \to Y_0$ be defined by the equality $Sy = (y,0)$. If $x(\cdot)\in W$ and $\|Fx(\cdot)-y\|_{l_2}\le\delta$, that is,

$$\sum_{j\in\mathbb{Z}}\nu_j|x_j|^2\le 1, \quad \sum_{j\in\mathbb{Z}}|x_j-y_j|^2\le\delta^2,$$

then obviously

$$\|I_0x(\cdot)-Sy\|^2_{Y_0} = (1-\widehat{\alpha})\delta^{-2}\sum_{j\in\mathbb{Z}}|x_j-y_j|^2 + \widehat{\alpha}\sum_{j\in\mathbb{Z}}\nu_j|x_j|^2\le 1.$$

According to Lemma 2 the squared value of the optimal recovery error does not exceed the value of the problem

$$\|Tx(\cdot)\|^2_{L_2(\mathbb{T})}\to\max, \quad \|I_0x(\cdot)\|^2_{Y_0}\le 1$$

which becomes exactly the problem (14) after transition to Fourier coefficients (by the Parseval equality) in the maximizing functional and changing $|x_j|^2$ by u_j .

3. Optimal method. It follows from Lemma 2 that an optimal method has the form $\widehat{\varphi} = T\circ\psi$ where Fourier coefficients $\{\psi_j\}_{j\in\mathbb{Z}}$ of the function $\psi=\psi(y)$ are solutions of the extremal problem

$$(1-\widehat{\alpha})\delta^{-2}\sum_{j\in\mathbb{Z}}|x_j-y_j|^2 + \widehat{\alpha}\sum_{j\in\mathbb{Z}}\nu_j|x_j|^2\to\min, \quad x(\cdot)\in X.$$

It is easy to see that

$$\psi_j = \frac{(1-\widehat{\alpha})\delta^{-2}}{(1-\widehat{\alpha})\delta^{-2}+\widehat{\alpha}\nu_j}y_j, \quad j\in\mathbb{Z}.$$

Substituting the expression for $\widehat{\alpha}$ we obtain the required result. □

The lower bound in (4) follows immediately from Lemma 1 and the optimality of the method (5) is verified directly.

Proof of Theorem 2. First of all, we note that the definition of the number s_0 by the equality (6) is well-defined since from the increase of the sequence $\{\nu_j\mu_j^{-1}\}_{j\in\mathbb{N}}$ follows the inequality

$$\frac{\mu_{N+1}-\mu_N}{\nu_{N+1}-\nu_N}\le\frac{\mu_{N+1}}{\nu_{N+1}}.$$

Thus $1\le s_0\le N$.

We turn our attention only to the solution of the extremal problem

$$\|Tx(\cdot)\|_{L_2(\mathbb{T})}\to\max, \quad \|Fx(\cdot)\|_{l_2^{2N+1}}\le\delta,\ x(\cdot)\in W, \tag{15}$$

because all other arguments largely repeat the proof of Theorem 1. Going over to Fourier coefficients and denoting the square of their moduli by u_j we arrive at the equivalent problem

$$(16)\qquad \sum_{j\in\mathbb{Z}}\mu_j u_j\to\max,\quad \sum_{|j|\le N}u_j\le\delta^2,\quad \sum_{j\in\mathbb{Z}}\nu_j u_j\le 1,\quad u_j\ge 0.$$

To solve this problem (just as to solve the problem (11)) it suffices to present such $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ and an admissible sequence $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ for which for all $u_j\ge 0$, $j\in\mathbb{Z}$,

$$(a_2)\quad \sum_{j\in\mathbb{Z}}(-\mu_j+\widehat{\lambda}_1\chi_j+\widehat{\lambda}_2\nu_j)u_j\ge\sum_{j\in\mathbb{Z}}(-\mu_j+\widehat{\lambda}_1\chi_j+\widehat{\lambda}_2\nu_j)\widehat{u}_j,$$

where

$$\chi_j=\begin{cases}1, & |j|\le N,\\ 0, & |j|>N,\end{cases}$$

and moreover,

$$(b_2)\quad \widehat{\lambda}_1\Bigg(\sum_{|j|\le N}\widehat{u}_j-\delta^2\Bigg)=0,\quad \widehat{\lambda}_2\Bigg(\sum_{j\in\mathbb{Z}}\nu_j\widehat{u}_j-1\Bigg)=0.$$

Let $\nu_{s+1}^{-1/2}\le\delta<\nu_s^{-1/2}$ and $1\le s\le s_0-1$. Define $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from the conditions

$$-\mu_s+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_s=0,$$
$$-\mu_{s+1}+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_{s+1}=0.$$

Then for $\widehat{\lambda}_1$ и $\widehat{\lambda}_2$ the equalities (12) are fulfilled from which it follows that $\widehat{\lambda}_1,\widehat{\lambda}_2>0$. In view of the fact that the sequence $\{-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j\}_{j\in\mathbb{Z}_+}$ has at most two sign changes we obtain

$$-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j\ge 0,\quad |j|\le N.$$

From the increase of the sequence $\{\nu_j\mu_j^{-1}\}_{j\in\mathbb{N}}$ and the fact that $s<s_0$ for $j\ge N+1$ we have

$$\frac{\mu_j}{\nu_j}\le\frac{\mu_{N+1}}{\nu_{N+1}}<\frac{\mu_{s+1}-\mu_s}{\nu_{s+1}-\nu_s}=\widehat{\lambda}_2.$$

Hence $-\mu_j+\widehat{\lambda}_2\nu_j>0$ for $|j|\ge N+1$. Defining $\widehat{u}_s$ and $\widehat{u}_{s+1}$ by the equalities (13) and putting $\widehat{u}_j=0$, $j\ne s,s+1$, we obtain that $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is an admissible sequence for which the conditions (a_2) and (b_2) are fulfilled. Consequently, $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is a solution of the problem (16).

Now let $0<\delta<\nu_{s_0}^{-1/2}$. Set

$$\widehat{\lambda}_1=\mu_{s_0}-\frac{\mu_{N+1}}{\nu_{N+1}}\nu_{s_0},\quad \widehat{\lambda}_2=\frac{\mu_{N+1}}{\nu_{N+1}}.$$

From the definition of s_0 we have

$$\frac{\mu_{s_0}-\mu_{s_0-1}}{\nu_{s_0}-\nu_{s_0-1}} > \frac{\mu_{N+1}}{\nu_{N+1}}.$$

Thus, $-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j \geq 0$ for $j=0, s_0-1$ and $-\mu_{s_0}+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_{s_0}=0$. From the condition on sign changes we obtain that $-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j \geq 0$ for all $|j| \leq s_0$. If $s_0 < N$, then by virtue of the definition of s_0

$$\frac{\mu_{s_0+1}-\mu_{s_0}}{\nu_{s_0+1}-\nu_{s_0}} \leq \frac{\mu_{N+1}}{\nu_{N+1}},$$

that is, $-\mu_{s_0+1}+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_{s_0+1} \geq 0$. Then from the condition on sign changes $-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j \geq 0$ for all $|j| \leq N$. If $|j| > N$, then

$$-\mu_j+\widehat{\lambda}_2\nu_j=\mu_j\frac{\mu_{N+1}}{\nu_{N+1}}\left(\frac{\nu_j}{\mu_j}-\frac{\nu_{N+1}}{\mu_{N+1}}\right) \geq 0.$$

Thus it is proved that for all $j \in \mathbb{Z}$

$$-\mu_j+\widehat{\lambda}_1\chi_j+\widehat{\lambda}_2\nu_j \geq 0.$$

Putting

$$\widehat{u}_{s_0}=\delta^2, \quad \widehat{u}_{N+1}=\frac{1-\delta^2\nu_{s_0}}{\nu_{N+1}}, \quad \widehat{u}_j=0,\ j \neq s_0, N+1,$$

it is easy to verify that $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is an admissible sequence for which the conditions (a_2) and (b_2) are fulfilled.

The case $\delta \geq \nu_1^{-1/2}$ is considered in the same way as in the proof of Theorem 1. □

For recovery of function itself from the class $W=W_2^r(\mathbb{T}), H_2^{r,\beta}(\mathbb{T})$, $A_2^{r,\beta}(\mathbb{T})$ we cannot apply formally Theorem 2 ($\mu_j=1$, $j \in \mathbb{Z}$, and hence the conditions of the theorem with respect to this sequence are not fulfilled). Nevertheless the scheme itself of the proof remains the same.

Proof of Theorem 3. 1. The lower bound. From Lemma 1 we obtain that the error of optimal recovery estimates from below by the value of the problem

$$\|Tx(\cdot)\|_{L_2(\mathbb{T})} \to \max, \quad |x_j| \leq \delta_j,\ |j| \leq N, \quad x(\cdot) \in W,$$

where $\{x_j\}_{|j|\leq N}$ are the first $2N+1$ of Fourier coefficients of the function $x(\cdot)$. Putting $u_j=|x_j|^2$, $j \in \mathbb{Z}$, we arrive at the equivalent problem

$$(17) \qquad \sum_{j\in\mathbb{Z}}\mu_j u_j \to \max, \quad \sum_{j\in\mathbb{Z}}\nu_j u_j \leq 1, \quad 0 \leq u_j \leq \delta_j^2,\ |j| \leq N.$$

To find a solution of this problem it suffices to find such $\widehat{\lambda} \geq 0$, $\widehat{\lambda}_j \geq 0$, $|j| \leq N$, and an admissible sequence $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ for which for all $u_j \geq 0$,

$j \in \mathbb{Z}$,

$$(a_3) \quad \sum_{j\in\mathbb{Z}}(-\mu_j + \widehat{\lambda}\nu_j + \widehat{\lambda}_j\chi_j)u_j \geq \sum_{j\in\mathbb{Z}}(-\mu_j + \widehat{\lambda}\nu_j + \widehat{\lambda}_j\chi_j)\widehat{u}_j,$$

and

$$(b_3) \quad \widehat{\lambda}\left(\sum_{j\in\mathbb{Z}}\nu_j\widehat{u}_j - 1\right) = 0, \quad \widehat{\lambda}_j(\widehat{u}_j - \delta_j^2) = 0, \ |j| \leq N.$$

Put $\widehat{\lambda} = \dfrac{\mu_{p_0+1}}{\nu_{p_0+1}}$,

$$\widehat{\lambda}_j = \begin{cases} \mu_j - \dfrac{\mu_{p_0+1}}{\nu_{p_0+1}}\nu_j, & |j| \leq p_0, \\ 0, & p_0 + 1 \leq |j| \leq N. \end{cases}$$

Define the sequence $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ by the equality

$$\widehat{u}_j = \begin{cases} \delta_j^2, & |j| \leq p_0, \\ \delta_j^2 \dfrac{1 - \sum_{|k|\leq p_0}\nu_k\delta_k^2}{(\delta_{p_0+1}^2 + \delta_{-p_0-1}^2)\nu_{p_0+1}}, & |j| = p_0 + 1, \\ 0, & |j| > p_0 + 1. \end{cases}$$

It follows from the definition of p_0 that the sequence $\widehat{u} = \{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is admissible. Moreover, $-\mu_j + \widehat{\lambda}\nu_j + \widehat{\lambda}_j = 0$ for $|j| \leq p_0$. By virtue of increase of the sequence $\{\nu_j\mu_j^{-1}\}_{j\in\mathbb{N}}$ for $|j| \geq p_0+1$ we have $-\mu_j+\widehat{\lambda}\nu_j \geq 0$. Thus the condition (a_3) is fulfilled. It is easily verified that the condition (b_3) is also fulfilled. Thus, $\widehat{u}$ is a solution of the problem (17). Substituting $\widehat{u}$ into the maximizing functional and extracting the square root we obtain

$$E(T, W^\nu, I_{\overline{\delta}}^{2N+1}) \geq \sqrt{\frac{\mu_{p_0+1}}{\nu_{p_0+1}} + \sum_{|j|\leq p_0}\left(\mu_j - \nu_j\frac{\mu_{p_0+1}}{\nu_{p_0+1}}\right)\delta_j^2}.$$

2. The upper bound. For the obtained above $\widehat{\lambda}$ and $\widehat{\lambda}_j$, $|j| \leq N$, set

$$\lambda = \widehat{\lambda} + \sum_{|j|\leq N}\widehat{\lambda}_j\delta_j^2, \quad \alpha = \frac{\widehat{\lambda}}{\lambda}, \quad \alpha_j = \frac{\widehat{\lambda}_j\delta_j^2}{\lambda}, \ |j| \leq N.$$

Then the condition (a_3) may be rewritten in the form

$$(a_4) \quad \sum_{j\in\mathbb{Z}}(-\mu_j + \lambda(\alpha\nu_j + \alpha_j\delta_j^{-2}\chi_j))u_j$$
$$\geq \sum_{j\in\mathbb{Z}}(-\mu_j + \lambda(\alpha\nu_j + \alpha_j\delta_j^{-2}\chi_j))\widehat{u}_j.$$

Since $\alpha + \sum\limits_{|j|\le N} \alpha_j = 1$ it is easy to verify that the condition

$$(b_4) \quad \alpha \sum_{j\in\mathbb{Z}} \nu_j \widehat{u}_j + \sum_{|j|\le N} \alpha_j \delta_j^{-2} \widehat{u}_j = 1$$

holds. The conditions (a_4) and (b_4) are sufficient in order that $\widehat{u}$ be a solution of the problem

$$(18) \qquad \sum_{j\in\mathbb{Z}} \mu_j u_j \to \max, \quad \alpha \sum_{j\in\mathbb{Z}} \nu_j u_j + \sum_{|j|\le N} \alpha_j \delta_j^{-2} u_j \le 1,\ u_j \ge 0.$$

Consequently, the values of the problems (17) and (18) coincide.

Let us use now Lemma 2. Let $Y_0 = l_2^{2N+1} \times l^\nu$. Define the semi-inner product on Y_0

$$((x,z),(x',z'))_{Y_0} = \sum_{|j|\le N} \alpha_j \delta_j^{-2} x_j \overline{x}'_j + \alpha \sum_{j\in\mathbb{Z}} \nu_j z_j \overline{z}'_j.$$

Define the operator $I_0\colon X \to Y_0$ by the equality

$$I_0 x(\cdot) = (\{x_j\}_{|j|\le N}, Fx(\cdot))$$

and define $S\colon l_\infty^{2N+1} \to Y_0$ by the equality $Sy = (y,0)$. If $x(\cdot) \in W$ and $|x_j - y_j| \le \delta_j$, $|j| \le N$, then

$$\|I_0 x(\cdot) - Sy\|_{Y_0}^2 = \alpha \sum_{j\in\mathbb{Z}} \nu_j |x_j|^2 + \sum_{|j|\le N} \alpha_j \delta_j^{-2} |x_j - y_j|^2 \le 1.$$

According to Lemma 2 the squared value of the optimal recovery error does not exceed the value of the problem

$$\|Tx(\cdot)\|_{L_2(\mathbb{T})}^2 \to \max, \quad \|I_0 x(\cdot)\|_{Y_0}^2 \le 1$$

which coincides with the problem (18) after going over to Fourier coefficients and changing $|x_j|^2$ by u_j. It remains to write out the optimal method of recovery which has the form $\widehat{\varphi} = T \circ \psi$ where the Fourier coefficients $\{\psi_j\}_{j\in\mathbb{Z}}$ of the function $\psi = \psi(y)$ are the solution of the extremal problem

$$\sum_{|j|\le N} \alpha_j \delta_j^{-2} |x_j - y_j|^2 + \alpha \sum_{j\in\mathbb{Z}} \nu_j |x_j|^2 \to \min, \quad x(\cdot) \in X.$$

It is easy to see that

$$\psi_j = \begin{cases} y_0, & j = 0, \\ \dfrac{\alpha_j \delta_j^{-2}}{\alpha \nu_j + \alpha_j \delta_j^{-2}} y_j, & 1 \le |j| \le p_0, \\ 0, & |j| > p_0. \end{cases}$$

Carrying out simple calculations related to substitution of expressions for α and α_j we obtain the required result. □

4. Some further results

4.1. **Recovery of functions defined on a sphere.** Let

$$\mathbb{S}^d = \left\{ x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : \sum_{j=1}^{d+1} x_j^2 = 1 \right\}$$

be the unit d-dimensional sphere. It is known (see [9]) that $L_2(\mathbb{S}^d) = \sum_{k=0}^{\infty} H_k$ where $\dim H_0 = n_0 = 1$,

$$\dim H_j = n_j = \frac{2j+d-1}{j}\binom{j+d-2}{j-1}, \quad j = 1, 2, \dots$$

(H_j is the set of spherical harmonic of order j). For the Laplace operator Δ and any $x(\cdot) \in H_j$ the equlity

$$\Delta x(\cdot) = -\Lambda_j x(\cdot)$$

holds, where $\Lambda_j = j(j+d-1)$. Let $\{Y_k^j\}_{k=1}^{n_j}$ is an orthonormal basis in H_j. For $\beta > 0$ the operator $(-\Delta)^{\beta/2}$ is defined by the equality

$$(-\Delta)^{\beta/2} x(\cdot) = \sum_{j \in \mathbb{N}} \Lambda_j^{\beta/2} \sum_{k=1}^{n_j} x_{jk} Y_k^j(\cdot),$$

where $x_{jk} = (x(\cdot), Y_k^j(\cdot))_{L_2(\mathbb{S}^d)}$ are the Fourier coefficients of the function $x(\cdot)$.

Set

$$W_2^\beta(\mathbb{S}^d) = \{\, x(\cdot) \in L_2(\mathbb{S}^d) : \|(-\Delta)^{\beta/2} x(\cdot)\|_{L_2(\mathbb{S}^d)} \le 1 \,\}.$$

Consider the problem of optimal recovery of the operator $T = (-\Delta)^{\gamma/2}$ on the class $W_2^\beta(\mathbb{S}^d)$ by the following information mappings:

1) the information $Ix(\cdot) = I_{\delta d} x(\cdot)$ about a function $x(\cdot) \in W_2^\beta(\mathbb{S}^d)$ is given as numbers y_{jk} such that

$$\sum_{j \in \mathbb{Z}_+} \sum_{k=1}^{n_j} |x_{jk} - y_{jk}|^2 \le \delta^2;$$

2) the information $Ix(\cdot) = I_{\delta d}^{N_m} x(\cdot)$ about a function $x(\cdot) \in W_2^\beta(\mathbb{S}^d)$ is given as numbers y_{jk}, $j = 0, 1, \dots, m$, $k = 1, \dots, n_j$, such that

$$\sum_{j=0}^{m} \sum_{k=1}^{n_j} |x_{jk} - y_{jk}|^2 \le \delta^2$$

(here $N_m = \sum_{j=0}^{m} n_j$);

3) the information $Ix(\cdot) = I_{\bar{\delta} d}^{N_m} x(\cdot)$ is such that there are known y_{jk}, $j = 0, 1, \dots, m$, $k = 1, \dots, n_j$, such that

$$|x_{jk} - y_{jk}| \le \delta_{jk}, \quad j = 0, 1, \dots, m, \ k = 1, \dots, n_j.$$

In the paper [6] it is shown that for all $\lambda_1, \lambda_2 \ge 0$ the function

$$f(t) = -(t(t+d-1))^\gamma + \lambda_1 + \lambda_2 (t(t+d-1))^\beta$$

vanishes at most at two points on the set $[0,+\infty)$. Hence it follows that for $\beta>\gamma>0$ for $\mu_j=\Lambda_j^\gamma$ and $\nu_j=\Lambda_j^\beta$ the conditions 1)–3) from section 2 are fulfilled.

Using the same scheme of the proof as in Theorems 1–3 we obtain their analogs for the considered problems. We turn our attention only to the statements of the corresponding results. In what follows we always assume that $\beta>\gamma>0$.

Theorem 4. *The eqality*

$$E((-\Delta)^{\gamma/2}, W_2^\beta(\mathbb{S}^d), I_{\delta d}) = \sqrt{\delta^2\Lambda_s^\gamma + (1-\delta^2\Lambda_s^\beta)\frac{\Lambda_{s+1}^\gamma-\Lambda_s^\gamma}{\Lambda_{s+1}^\beta-\Lambda_s^\beta}}$$

holds if $\Lambda_{s+1}^{-\beta/2}\le\delta<\Lambda_s^{-\beta/2}$, $s\ge1$, *and*

$$E((-\Delta)^{\gamma/2}, W_2^\beta(\mathbb{S}^d), I_{\delta d}) = d^{(\gamma-\beta)/2},$$

if $\delta\ge\Lambda_1^{-\beta/2}$. *Moreover, the method*

$$\widehat{\varphi}(y)(\cdot)=\sum_{j\in\mathbb{N}}\Lambda_j^{\gamma/2}\left(1+\Lambda_j^\beta\frac{\Lambda_{s+1}^\gamma-\Lambda_s^\gamma}{\Lambda_s^\gamma\Lambda_{s+1}^\beta-\Lambda_{s+1}^\gamma\Lambda_s^\beta}\right)^{-1}\sum_{k=1}^{n_j}y_{jk}Y_k^j(\cdot)$$

is optimal if $\Lambda_{s+1}^{-\beta/2}\le\delta<\Lambda_s^{-\beta/2}$, $s\ge1$, *and if* $\delta\ge\Lambda_1^{-\beta/2}$, *then* $\widehat{\varphi}(y)(\cdot)=0$ *is an optimal method.*

Put

$$s_0=s_0(m)=\min\left\{s\in\mathbb{N}:\frac{\Lambda_{s+1}^\gamma-\Lambda_s^\gamma}{\Lambda_{s+1}^\beta-\Lambda_s^\beta}\le\Lambda_{m+1}^{\gamma-\beta}\right\}.$$

Theorem 5. *For* $\delta\ge\Lambda_{s_0}^{-\beta/2}$

$$E((-\Delta)^{\gamma/2}, W_2^\beta(\mathbb{S}^d), I_{\delta d}^{N_m}) = E((-\Delta)^{\gamma/2}, W_2^\beta(\mathbb{S}^d), I_{\delta d})$$

and the method

$$\widehat{\varphi}(y)(\cdot)=\sum_{j=1}^{m}\Lambda_j^{\gamma/2}\left(1+\Lambda_j^\beta\frac{\Lambda_{s+1}^\gamma-\Lambda_s^\gamma}{\Lambda_s^\gamma\Lambda_{s+1}^\beta-\Lambda_{s+1}^\gamma\Lambda_s^\beta}\right)^{-1}\sum_{k=1}^{n_j}y_{jk}Y_k^j(\cdot)$$

is optimal if $\Lambda_{s+1}^{-\beta/2}\le\delta<\Lambda_s^{-\beta/2}$, $1\le s\le s_0-1$, *and if* $\delta\ge\Lambda_1^{-\beta/2}$, *then* $\widehat{\varphi}(y)(\cdot)=0$ *is an optimal method. For* $0<\delta<\Lambda_{s_0}^{-\beta/2}$

$$E((-\Delta)^{\gamma/2}, W_2^\beta(\mathbb{S}^d), I_{\delta d}^{N_m}) = \sqrt{\delta^2\Lambda_{s_0}^\gamma+(1-\delta^2\Lambda_{s_0}^\beta)\Lambda_{m+1}^{\gamma-\beta}}$$

and

$$\widehat{\varphi}(y)(\cdot)=\sum_{j=1}^{m}\Lambda_j^{\gamma/2}\left(1+\Lambda_j^\beta\frac{\Lambda_{m+1}^\gamma}{\Lambda_{s_0}^\gamma\Lambda_{m+1}^\beta-\Lambda_{m+1}^\gamma\Lambda_{s_0}^\beta}\right)^{-1}\sum_{k=1}^{n_j}y_{jk}Y_k^j(\cdot)$$

is an optimal method.

Put

$$p_0 = p_0(\overline{\delta}) = \max\Big\{\, p \in \mathbb{Z}_+ : 1 - \sum_{j=0}^{p} \Lambda_j^{\beta} \sum_{k=1}^{n_j} \delta_{jk}^2 > 0,\ 0 \le p \le m \,\Big\}.$$

Theorem 6. *The equality*

$$E((-\Delta)^{\gamma/2}, W_2^{\beta}(\mathbb{S}^d), I_{\overline{\delta}d}^{N_m}) = \sqrt{\Lambda_{p_0+1}^{\gamma-\beta} + \sum_{j=1}^{p_0} (\Lambda_j^{\gamma} - \Lambda_j^{\beta}\Lambda_{p_0+1}^{\gamma-\beta}) \sum_{k=1}^{n_j} \delta_{jk}^2}$$

holds, moreover, the method

$$\widehat{\varphi}(y)(\cdot) = \sum_{j=1}^{p_0} \Lambda_j^{\gamma/2} \left(1 - \left(\frac{\Lambda_{p_0+1}}{\Lambda_j}\right)^{\gamma-\beta}\right) \sum_{k=1}^{n_j} y_{jk} Y_k^j(\cdot)$$

is optimal.

4.2. **The Hardy–Sobolev and Bergman–Sobolev classes on the unit disk.** Denote by $\mathcal{H}_2(D)$ the space of functions $x(\cdot)$ analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the condition

$$\|x(\cdot)\|_{\mathcal{H}_2(D)} = \sup_{0<\rho<1} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |x(\rho e^{it})|^2\, dt\right)^{1/2} < \infty.$$

The *Hardy–Sobolev class* $H_2^r(D)$ is the set of functions $x(\cdot)$ analytic in D for which $\|x^{(r)}(\cdot)\|_{\mathcal{H}_2(D)} \le 1$.

Denote by $\mathcal{A}_2(D)$ the space of functions $x(\cdot)$ analytic in the unit disk D and satisfying the condition

$$\|x(\cdot)\|_{\mathcal{A}_2(D)} = \left(\int_D |x(z)|^2\, d\sigma(z)\right)^{1/2} < \infty,$$

where σ is the plane Lebesgue measure. The *Bergman–Sobolev class* $A_2^r(D)$ is the set of functions $x(\cdot)$ analytic in D for which $\|x^{(r)}(\cdot)\|_{\mathcal{A}_2(D)} \le 1$.

Consider the problem of optimal recovery of the k-th derivative on the classes $W = H_2^r$ and A_2^r by the information about coefficients of function power series given with the error δ in the norm of the space l_2. In other words, we assume that for every function $x(\cdot) \in W$ such that

$$x(z) = \sum_{j\in\mathbb{Z}_+} a_j z^j$$

the numbers $\{y_j\}_{j\in\mathbb{Z}_+}$ are known such that

$$\sum_{j\in\mathbb{Z}_+} |a_j - y_j|^2 \le \delta^2.$$

We denote by I_δ^+ the corresponding information mapping. The studied problem of optimal recovery is written in the form

$$E(D^k, W, I_\delta^+) = \inf_{\varphi\colon l_2\to\mathcal{W}} \sup_{\substack{x(\cdot)\in W,\ y\in l_2\\ \|F^+x(\cdot)-y\|_{l_2}\le\delta}} \|x^{(k)}(\cdot)-\varphi(y)(\cdot)\|_{\mathcal{W}},$$

where $\mathcal{W} = \mathcal{H}_2(D), \mathcal{A}_2(D)$ and $F^+x(\cdot) = \{a_j\}_{j\in\mathbb{Z}_+}$ are the coefficients of the power series of the function $x(\cdot)$.

We formulate the analogue of Theorem 1 which may be obtained by the scheme described above. Set for a fixed k and r $(1\le k<r)$

$$\mu_j(W) = \begin{cases} \left(\dfrac{j!}{(j-k)!}\right)^2, & j\ge k,\ W = H_2^r(D),\\ \left(\dfrac{j!}{(j-k)!}\right)^2 \dfrac{1}{j-k+1}, & j\ge k,\ W = A_2^r(D),\\ 0, & j<k,\end{cases}$$

$$\nu_j(W) = \begin{cases} \left(\dfrac{j!}{(j-r)!}\right)^2, & j\ge r,\ W = H_2^r(D),\\ \left(\dfrac{j!}{(j-r)!}\right)^2 \dfrac{1}{j-r+1}, & j\ge r,\ W = A_2^r(D),\\ 0, & j<r.\end{cases}$$

Theorem 7. *For $W = H_2^r$ or A_2^r and all $1\le k<r$ for $\delta\ge\nu_r^{-1/2}(W)$ the equality*

$$E(D^k, W, I_\delta^+) = \sqrt{\delta^2\mu_{r-1}(W) + \frac{\mu_r(W)-\mu_{r-1}(W)}{\nu_r(W)}}$$

holds and for $\nu_{s+1}^{-1/2}(W)\le\delta<\nu_s^{-1/2}(W)$, $s\ge r$, the equality

$$E(D^k, W, I_\delta^+) = \sqrt{\delta^2\mu_s(W) + (1-\delta^2\nu_s(W))\frac{\mu_{s+1}(W)-\mu_s(W)}{\nu_{s+1}(W)-\nu_s(W)}}$$

holds. Moreover, the method

$$\widehat{\varphi}(y)(z) = \sum_{j=k}^{\infty}\left(1+\nu_j(W)\frac{\mu_{s+1}(W)-\mu_s(W)}{\mu_s(W)\nu_{s+1}(W)-\mu_{s+1}(W)\nu_s(W)}\right)^{-1} y_j z^{j-k}$$

is optimal if $\nu_{s+1}^{-1/2}(W)\le\delta<\nu_s^{-1/2}(W)$, $s\ge r$ and if $\delta\ge\nu_r^{-1/2}(W)$, then $\widehat{\varphi}(y)(z)\equiv 0$ is an optimal method.

The cases when only a finite number of coefficients of power series is known with an error in the mean square or uniform metric may be considered just in the same way as it was done for the periodic case.

References

[1] *Micchelli C. A., Rivlin T. J.* A survey of optimal recovery // Optimal estimation in approximation theory / ed. C. A. Micchelli, T. J. Rivlin. New York: Plenum Press, 1977. P. 1–54.

[2] *Melkman A. A., Micchelli C. A.* Optimal estimation of linear operators in Hilbert spaces from inaccurate data // SIAM J. Numer. Anal. 1979. V. 16. P. 87–105.

[3] *Micchelli C. A., Rivlin T. J.* Lectures on optimal recovery // Numerical analysis, Proc. SERC Summer Sch., Lancaster/Engl., 1984. Berlin: Springer–Verlag, 1984. P. 21–93. (Lecture Notes in Math. V. 1129.)

[4] *Micchelli C. A.* Optimal estimation of linear operators in Hilbert spaces from inaccurate data: a second look // Numer. Algorithms. 1993. V. 5. P. 375–390.

[5] *Plaskota L.* Noisy information and computational complexity. Cambridge: Cambridge Univ. Press, 1996.

[6] *Magaril-Il'yaev G. G., Tikhomirov V. M.* Kolmogorov-type inequalities for derivatives // Mat. Sb. 1997. V. 188. №12. P. 73–106; English transl. in Sbornic: Mathematics. 1997. V. 188. №12. P. 1799–1832.

[7] *Magaril-Il'yaev G. G., Tikhomirov V. M.* Convex Analysis and its Applications, Editorial URSS, Moscow, 2000.

[8] *Magaril-Il'yaev G. G., Osipenko K. Yu., Tikhomirov V. M.* Optimal reconstruction and the theory of extremum // Dokl. Ross. Akad. Nauk. 2001. V. 379. №2. P. 161–164; English transl. in Doklady Mathematics. 2001. V. 64. №1. P. 32–35.

[9] *Stein E. M., Weiss G.* Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton, 1971.

Moscow State Institute of Radio Engineering, Electronics and Automation (Technology University)

MATI — Russian State Technological University