

On Exact Values of n -Widths in a Hilbert Space*

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There are two basic classes of functions for which the exact values of Kolmogorov n -widths have been calculated. They are, on the one hand, classes of real functions defined by variation diminishing kernels and similar classes of analytic functions, and, on the other hand, classes of functions in a Hilbert space which are elliptical cylinders or generalized octahedra (see [1, 2]). This paper is devoted to a survey and to new results in this second case. For n -widths of ellipsoids, elliptic cylinders, and generalized octahedra upper bounds for the n -widths are based on the Fourier method. The lower bounds are based on the method of “embedded balls” (see [1]) for ellipsoids, and the method of averaging for generalized octahedra. General theorems concerning elliptical cylinders and generalized octahedra are proved in Section 1 and Section 2. In Section 3 various corollaries from these general theorems are considered. Some additional problems (average n -widths, extremal spaces for an ellipsoids and octahedra, etc) are discussed in Section 4.

*This research was supported in part by the Russian Foundation for Basic Research (Grants #96-01-10035 and #99-01-01181), the State Program on Support of Leading Scientific Schools (Grant #96-15-96072), and INTAS-OPEN #97-1050.

INTRODUCTION

Let H be a Hilbert space and $C \subset H$ a centrally symmetric set. For $n \in \mathbb{Z}_+$ the *Kolmogorov n -widths* of C in H are given by

$$d_n(C, H) = \inf_{L_n} \sup_{x \in C} \inf_{y \in L_n} \|x - y\|_H,$$

where the left-most infimum is taken over all n -dimensional linear subspaces L_n of H .

Let $r \in \mathbb{N}$ and $1 \leq p \leq \infty$. Denote by $W_p^r(\mathbb{T})$ ($\mathbb{T} = [0, 2\pi)$) the Sobolev class of 2π -periodic functions whose $(r-1)$ -st derivative is absolutely continuous and

$$\|x^{(r)}\|_{L_p(\mathbb{T})} := \left(\int_{\mathbb{T}} |x^{(r)}(t)|^p dt \right)^{1/p} \leq 1.$$

The subject considered in this paper goes back to two papers by Kolmogorov. In [3] Kolmogorov proved the formula

$$d_{2n-1}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = d_{2n}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = n^{-r}.$$

In the paper of Kolmogorov, Petrov, Smirnov [4], which was supplemented by Maltsev [5], the equality

$$d_n(Bl_1^N, l_2^N) = \sqrt{\frac{N-n}{N}}, \tag{1}$$

where

$$l_p^N := \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \|x\|_{l_p^N}^p := \sum_{k=1}^N |x_k|^p \right\}, \quad 1 \leq p < \infty,$$

and Bl_1^N is the unit ball in l_1^N was, in fact, proved. The authors of these papers did not actually state, that they had calculated n -widths. This was noted by Stechkin [6]). Note that the Sobolev class $W_2^r(\mathbb{T})$ is an elliptical cylinder (it is the orthogonal sum of the one-dimensional space of constants and compact ellipsoid) and Bl_1^N is a regular octahedra in \mathbb{R}^N . We consider generalizations of these two results.

1. n -WIDTHS OF ELLIPSOIDS AND ELLIPTICAL CYLINDERS

An *ellipsoid* is the image of a ball of a Hilbert space under a linear continuous mapping. If H and H_1 are Hilbert spaces, BH is the unit ball in H , and $T: H \rightarrow H_1$ is a linear continuous operator, then $T(BH)$ is an ellipsoid which we denote

by $E(T)$. Let L be a finite-dimensional subspace in H_1 . We call the orthogonal sum of an ellipsoid $E(T)$ and L

$$E(T) \oplus L = \{y = y_1 + y_2 \in H_1 \mid y_1 \in E(T), y_2 \in L, y_1 \perp y_2\}$$

an *elliptical cylinder with base $E(T)$ and generalized axis L* .

Denote by N , $0 \leq N \leq \infty$ the dimension of $\text{span } E(T)$. Further we calculate the n -widths of compact ellipsoids and elliptical cylinders with compact base.

Let T be a compact operator. By the Hilbert–Schmidt theorem (see, for example, [7, page 231]) for the self-adjoint compact operator $T'T$ (T' is the adjoint operator to T) there exists an orthonormal system of eigenvectors $\{e_k\}_{k \geq 1}$ which corresponding to eigenvalues s_k^2 , $s_k \downarrow 0$, $s_k \neq 0$, such that each element $x \in H$ has the unique representation

$$x = \sum_{k \geq 1} \langle x, e_k \rangle e_k + \xi, \quad (2)$$

where $\xi \in \text{Ker } T$. (The numbers s_k are called the s -numbers of T .)

The following theorem was proved by many authors (see Comments below).

THEOREM 1 (n -WIDTHS OF COMPACT ELLIPSOIDS). *Let H and H_1 be Hilbert spaces, $T: H \rightarrow H_1$ a compact operator, $C = E(T) \oplus L_m$ ($\dim E(T) = N$, $\dim L_m = m$) and $n \in \mathbb{Z}_+$. Then*

$$d_n(C, H_1) = \begin{cases} \infty, & n < m, \\ s_{n+1-m}, & m \leq n \leq N + m, \\ 0, & n > N + m. \end{cases}$$

The linear, Gel'fand, and Bernstein n -widths satisfy the same equalities.

Proof. We prove the theorem for the Kolmogorov n -width. The statement of the theorem for the cases when $n < m$ and $n > N + m$ may be easily checked. Let $n < N$ and for simplicity assume $m = 0$ (the general case easily follows from this). The upper bound will be proved by the *Fourier method*. Let $y \in E(T)$ that is $y = Tx$, $\|x\|_H \leq 1$. By (2) $y = \sum_{k \geq 1} \langle x, e_k \rangle T e_k$ and $\|x\|_H^2 = \sum_{k \geq 1} |\langle x, e_k \rangle|^2 + \|\xi\|_H^2 \leq 1$. Let us approximate y by $S_n y = \sum_{k=1}^n \langle x, e_k \rangle T e_k$. Then taking into account the orthogonality of the system $\{T e_k\}$, we have

$$\|y - S_n y\|_{H_1}^2 = \sum_{k \geq n+1} s_k^2 |\langle x, e_k \rangle|^2 \leq s_{n+1}^2 \sum_{k \geq n+1} |\langle x, e_k \rangle|^2 \leq s_{n+1}^2.$$

The upper bound is proved. The lower bound will be proved by the method of *embedded balls*. We consider the $n+1$ -dimensional subspace $\widehat{L} = \text{span}\{T e_k\}_{k=1}^{n+1}$ of H_1 and show that the set $s_{n+1} B H_1 \cap \widehat{L}$ lies in $E(T)$. Let $y \in s_{n+1} B H_1 \cap \widehat{L}$.

Then $y = \sum_{k=1}^{n+1} y_k T e_k$ and $\|y\|_{H_1}^2 = \sum_{k=1}^{n+1} y_k^2 s_k^2 \leq s_{n+1}^2$. If $x = \sum_{k=1}^{n+1} y_k e_k$, then it is clear that $y = Tx$ and since

$$\|x\|_H^2 = \sum_{k=1}^{n+1} y_k^2 = \sum_{k=1}^{n+1} \frac{y_k^2 s_k^2}{s_k^2} \leq \frac{1}{s_{n+1}} \sum_{k=1}^{n+1} y_k^2 s_k^2 \leq 1,$$

we obtain that $s_{n+1} B H_1 \cap \hat{L} \subset E(T)$. By the theorem on n -widths of a ball (see, for example, [2, p. 12]) which is trivial for a Hilbert space, we have $d_n(E(T), H_1) \geq d_n(s_{n+1} B H_1 \cap \hat{L}, H_1) = s_{n+1}$. ■

2. n -WIDTHS OF GENERALIZED OCTAHEDRA

In finite-dimensional geometry an octahedra is the convex hull of a simplex with a vertex at the origin and simplex symmetric to it. For octahedra which are so defined there is no known general method for calculating its n -widths. But it is possible to calculate n -widths for octahedra in \mathbb{R}^N which are the convex hull of the vectors $\{\pm f_k\}$, $1 \leq k \leq N$ obtained from one vector $K = (k_1, \dots, k_N)$ by cyclical permutation. Such octahedra may be considered as Sobolev classes $W_1^K(\mathbb{Z}_N)$ consisting of functions $y = (y_1, \dots, y_N)$ on the cyclical group of order N defined by a convolution

$$W_1^K(\mathbb{Z}_N) = \{y \in \mathbb{R}^N \mid y = K * x, \|x\|_{l_1^N} \leq 1\},$$

where $x = (x_1, \dots, x_N)$, $y_i = \sum_{j=1}^N k_{i+j-1} x_j$, and the summation is carried out modulo N . The regular octahedra can be defined in this same way as a convolution class on the cyclical group \mathbb{Z}_N with the kernel K equal to 1 at the zero of the group and 0 at all other elements.

It gives us the possibility to consider generalized Sobolev classes of so-called sourcewise represented functions which are similar to generalized octahedra.

First we recall some definitions (details may be found in [8]). Let G be a compact group with Haar measure μ ($\mu(G) = 1$). Let $\{T^\alpha\}_{\alpha \in \mathcal{A}}$ (where \mathcal{A} is at most a countable set) be a complete system of finite-dimensional nonreducible unitary representations of G . For each $\alpha \in \mathcal{A}$ we denote by $t_{ij}^\alpha(\cdot)$, $i, j = 1, \dots, n_\alpha = \dim T^\alpha$, the matrix elements of the representation T^α in some orthonormal basis. These functions are continuous and the functions $\{\sqrt{n_\alpha} t_{ij}^\alpha(\cdot)\}$, $\alpha \in \mathcal{A}$, $i, j = 1, \dots, n_\alpha$ form an orthonormal basis in $L_2(G)$. Note that if G is an Abelian group, then all representations T^α , $\alpha \in \mathcal{A}$, are one-dimensional. For each $\alpha \in \mathcal{A}$ and $1 \leq j \leq n_\alpha$ set $H_j^\alpha = \text{span}\{t_{ij}^\alpha(\cdot) \mid i = 1, \dots, n_\alpha\}$. The space $L_2(G)$ is represented as the direct sum of those spaces which are left translation-invariant.

A set X is called a *homogeneous G -space* if the group G acts transitively on X . In other words, if there exists a map $G \times X \rightarrow X$, $(g, x) \rightarrow gx$ such that $(g_2 g_1)x = g_2(g_1 x)$, $ex = x$ (e is the unity element of G) for all $g_1, g_2 \in G$ and $x \in X$ and in addition for every $x_1, x_2 \in X$ there exists a $g \in G$ for which

$x_2 = gx_1$. It is obvious that any group G is a homogeneous G -space with respect to the operation $(g, g_0) \rightarrow gg_0$.

Let $x_0 \in X$, $H = \{g \in G \mid gx_0 = x_0\}$ and G/H be the set of (left) residue classes of group G on the subgroup H . Consider the map $p: X \rightarrow G/H$ which associates x with the residue class gH such that $gx_0 = x$. The map p is a one-to-one mapping. Thus any function on X may be considered as a function on G , which is constant on the residue classes. By virtue of this fact for any topological homogeneous G -space X with compact group G and measure ν invariant with respect to G (that is $\nu(A) = \nu(gA)$ for any measurable subset $A \subset X$ and $g \in G$) the structure of $L_2(X)$ is analogous to the structure of $L_2(G)$. More precisely $L_2(X)$ is a direct sum of at most an enumerable set of finite-dimensional subspaces H_k invariant with respect to G consisting of continuous functions.

We will need the following auxiliary result.

LEMMA 1. *Let X be a topological homogeneous G -space with compact group G and probability measure invariant with respect to G . If $\{e_k(\cdot)\}_{k=1}^n$ is an orthonormal system of continuous functions from $L_2(X)$ such that $L_n = \text{span}\{e_k(\cdot)\}_{k=1}^n$ is invariant with respect to G , then*

$$\sum_{k=1}^n |e_k(\cdot)|^2 \equiv n.$$

Proof. For $x \in X$ consider the function

$$\xi_x(\cdot) = \sum_{k=1}^n \overline{e_k(x)} e_k(\cdot).$$

If $y(\cdot) \in L_n$ then it is clear that

$$\langle y(\cdot), \xi_x(\cdot) \rangle = y(x). \quad (3)$$

Let $x_1, x_2 \in X$ and $g \in G$ such that

$$x_2 = gx_1. \quad (4)$$

Using the invariance of the measure, (3), and (4), for every $y(\cdot) \in L_n$ we have

$$\langle y(\cdot), \xi_{x_2}(g\cdot) \rangle = \langle y(g^{-1}\cdot), \xi_{x_2}(\cdot) \rangle = y(g^{-1}x_2) = y(x_1) = \langle y(\cdot), \xi_{x_1}(\cdot) \rangle.$$

Thus $\xi_{x_2}(g\cdot) = \xi_{x_1}(\cdot)$. Substituting here x_1 we obtain that

$$\sum_{k=1}^n |e_k(x_1)|^2 = \sum_{k=1}^n |e_k(x_2)|^2 = C.$$

Since the given measure is a probability measure and the system $\{e_k(\cdot)\}_{k=1}^n$ is orthonormal we have

$$C = \int_X \sum_{k=1}^n |e_k(x)|^2 d\mu(x) = n. \quad \blacksquare$$

Let X be a topological homogeneous G -space with compact group G and probability measure invariant with respect to G . As was mentioned, in this case $L_2(X)$ may be represented in the form

$$L_2(X) = \sum_{k \geq 1} H_k, \quad \dim H_k = n_k < \infty,$$

where the H_k are shift-invariant spaces of continuous functions. Consider the classes of functions represented as a convolution with kernels

$$K(t, \tau) = \sum_{k \geq 1} \sum_{j=1}^{n_k} \gamma_{kj} e_{kj}(t) \overline{e_{kj}(\tau)}, \quad (5)$$

where the $\{e_{kj}(\cdot)\}_{j=1}^{n_k}$ is an orthonormal basis for H_k of continuous functions and $\gamma_{kj} \in \mathbb{C}$ such that

$$\sum_{k \geq 1} n_k \max_{1 \leq j \leq n_k} |\gamma_{kj}|^2 < \infty. \quad (6)$$

The function $K(\cdot, \cdot)$ induces the operator

$$Ax(t) = \int_X K(t, \tau) x(\tau) d\mu(\tau).$$

We show that A is a continuous operator from $L_1(X)$ into $L_2(X)$. Indeed, by the Cauchy–Bunyakovsky inequality for all $t \in X$

$$\begin{aligned} |Ax(t)| &\leq \int_X (|K(t, \tau)|^2 |x(\tau)|)^{1/2} |x(\tau)|^{1/2} d\mu(\tau) \\ &\leq \left(\int_X |K(t, \tau)|^2 |x(\tau)| d\mu(\tau) \right)^{1/2} \|x(\cdot)\|_{L_1(X)}^{1/2}. \end{aligned}$$

Squaring this inequality, integrating it, and then changing the order of integration, we obtain

$$\begin{aligned} \|Ax(\cdot)\|_{L_2(X)}^2 &\leq \|x(\cdot)\|_{L_1(X)}^2 \sup_{t \in X} \int_X |K(t, \tau)|^2 d\mu(\tau) \\ &= \|x(\cdot)\|_{L_1(X)}^2 \sup_{t \in X} \|K(t, \cdot)\|_{L_2(X)}^2. \end{aligned}$$

According to the Parseval equality for all $t \in X$

$$\|K(t, \cdot)\|_{L_2(X)}^2 = \sum_{k \geq 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2 |e_{kj}(t)|^2.$$

By Lemma 1 $\sum_{j=1}^{n_k} |e_{kj}(t)|^2 = n_k$. In view of (6) we have that $\|K(t, \cdot)\|_{L_2(X)} \leq C$. Therefore $A: L_1(X) \rightarrow L_2(X)$ is a continuous operator.

Set

$$W_1^K(X) = \{y(\cdot) \in L_2(X) \mid y(\cdot) = Ax(\cdot), \|x(\cdot)\|_{L_1(X)} \leq 1\}.$$

THEOREM 2. *Let X be a topological homogeneous G -space with compact group G and probability measure invariant with respect to G . Let $K: X \times X \rightarrow \mathbb{C}$ be a function of the form (5), where the γ_{kj} satisfy the additional condition $|\gamma_{kj}| = \lambda_k$, $1 \leq j \leq n_k$, $k \geq 1$. Assume that $\{\lambda_k\}_{k \geq 1}$ are in decreasing order. Then for all $n = n_1 + \dots + n_s$*

$$d_n(W_1^K(X), L_2(X)) = \left(\sum_{k \geq s+1} \lambda_k^2 n_k \right)^{1/2}.$$

Proof. Since $W_1^K(X) = \text{clco}\{K(\cdot, \tau)\}_{\tau \in X}$ it is sufficient to prove the statement of the theorem for the set $\{K(\cdot, \tau)\}_{\tau \in X}$.

The upper bound. We use the Fourier method projecting our class on the subspace $L_n = \text{span}\{e_{kj}(\cdot) \mid 1 \leq j \leq n_k, 1 \leq k \leq s\}$. Then for any $\tau \in X$ using the Parseval equality, hypothesis of the theorem, and Lemma, we have

$$d^2(K(\cdot, \tau), L_n, L_2(X)) = \sum_{k \geq s+1} \lambda_k^2 n_k.$$

Hence the required estimate follows.

The lower bound. We use the method of averaging. Let L_n be an n -dimensional subspace of $L_2(X)$, and $\{f_m\}_{m=1}^n$ an orthonormal basis of L_n . Then for all $\tau \in X$

$$d^2(K(\cdot, \tau), L_n, L_2(X)) = \|K(\cdot, \tau)\|_{L_2(X)}^2 - \sum_{m=1}^n \left| \int_X K(t, \tau) \overline{f_m(t)} d\mu(t) \right|^2. \quad (7)$$

In view of the hypothesis of the theorem and by Lemma 1 we have

$$\|K(\cdot, \tau)\|_{L_2(X)}^2 = \sum_{k \geq 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2 |e_{kj}(t)|^2 = \sum_{k \geq 1} \lambda_k^2 \sum_{j=1}^{n_k} |e_{kj}(t)|^2 = \sum_{k \geq 1} \lambda_k^2 n_k. \quad (8)$$

Furthermore,

$$\left| \int_X K(t, \tau) \overline{f_m(t)} d\mu(t) \right|^2 = \left| \sum_{k \geq 1} \sum_{j=1}^{n_k} \gamma_{kj} \overline{e_{kj}(\tau)} \int_X e_{kj}(t) \overline{f_m(t)} d\mu(t) \right|^2.$$

Substituting it and (8) into (7), integrating the obtained expression and using the Parseval equality with the hypothesis of the theorem, we obtain

$$\begin{aligned} & \int_X d^2(K(\cdot, \tau), L_n, L_2(X)) d\mu(\tau) \\ &= \sum_{k \geq 1} \lambda_k^2 n_k - \sum_{m=1}^n \sum_{k \geq 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2 \left| \int_X e_{kj}(t) \overline{f_m(t)} d\mu(t) \right|^2 \\ &= \sum_{k \geq 1} \lambda_k^2 n_k - \sum_{m=1}^n \sum_{k \geq 1} \lambda_k^2 \sum_{j=1}^{n_k} \left| \int_X e_{kj}(t) \overline{f_m(t)} d\mu(t) \right|^2. \quad (9) \end{aligned}$$

For $k \geq 1$ set

$$\hat{c}_k = \sum_{m=1}^n \sum_{j=1}^{n_k} \left| \int_X e_{kj}(t) \overline{f_m(t)} d\mu(t) \right|^2.$$

It is easy to check that $0 \leq \hat{c}_k \leq n_k$ and $\sum_{k \geq 1} \hat{c}_k = n_1 + \dots + n_s$. Consider the problem of linear programming

$$\sum_{k \geq 1} \lambda_k^2 c_k \rightarrow \max, \quad 0 \leq c_k \leq n_k, \quad \sum_{k \geq 1} c_k = n_1 + \dots + n_s.$$

The solution of this problem is evidently

$$c_k = n_k, \quad 1 \leq k \leq s, \quad c_k = 0, \quad k \geq s+1$$

(we recall that $\lambda_1 \geq \lambda_2 \geq \dots$). Thus we obtain the lower bound for the left-hand side of (9)

$$c_k = n_k, \quad 1 \leq k \leq s, \quad c_k = 0, \quad k \geq s+1.$$

Standard arguments now lead to the required estimate. \blacksquare

3. COROLLARIES FROM THE GENERAL THEOREMS

We begin with n -widths of convolution classes of functions defined on a compact group. Let G be a compact group and $K(\cdot) \in L_2(G)$. The operator of convolution is defined as

$$Tx(g) = \int_G K(gs^{-1})x(s) d\mu(s). \quad (10)$$

It is a compact operator from $L_2(G)$ into $L_2(G)$. Moreover, it follows from the Minkowski inequality that (10) is a continuous operator from $L_1(G)$ into $L_2(G)$. Set

$$W_p^K(G) = \left\{ y(\cdot) \in L_2(G) \mid y(\cdot) = Tx(\cdot), \quad \|x(\cdot)\|_{L_p(G)} \leq 1 \right\}, \quad p = 1, 2.$$

THEOREM 3. *Let G be a compact group and $K(\cdot) \in L_2(G)$ be such that its Fourier coefficients c_{ij}^α when expanded in the orthonormal basis $e_{ij}^\alpha(\cdot) = \sqrt{n_\alpha} t_{ij}^\alpha(\cdot)$, $\alpha \in \mathcal{A}$, $i, j = 1, \dots, n_\alpha$, satisfy the condition: for any $\alpha \in \mathcal{A}$ the matrix $C_\alpha = (c_{ij}^\alpha)_{i,j=1}^{n_\alpha}$ has the form $\lambda_\alpha U_\alpha$ where $\lambda_\alpha \in \mathbb{C}$ and U_α is a unitary matrix. Assume that $\{\lambda_k / \sqrt{n_k}\}_{k \geq 1}$ is the sequence $\{\lambda_\alpha / \sqrt{n_\alpha}\}_{\alpha \in \mathcal{A}}$ ordered in decreasing order. Then for all $m \geq 1$ ($n_0 = 0$)*

1) *for any n such that $n_1^2 + \dots + n_{m-1}^2 < n \leq n_1^2 + \dots + n_m^2$*

$$d_n(W_2^K(G), L_2(G)) = \frac{|\lambda_m|}{\sqrt{n_m}},$$

2) *for any $n = n_1^2 + \dots + n_{m-1}^2 + n_m s_m$, where $1 \leq s_m \leq n_m$,*

$$d_n(W_1^K(G), L_2(G)) = \left(|\lambda_m|^2 (n_m - s_m) + \sum_{k \geq m+1} |\lambda_k|^2 n_k \right)^{1/2}.$$

Proof. 1. Using properties of the matrix elements

$$t_{ij}^\alpha(g_1 g_2) = \sum_{k=1}^{n_\alpha} t_{ik}^\alpha(g_1) t_{kj}^\alpha(g_2), \quad t_{ij}^\alpha(g) = \overline{t_{ji}^\alpha(g^{-1})},$$

we have

$$K(g s^{-1}) = \sum_{\alpha \in \mathcal{A}} \sum_{i,j=1}^{n_\alpha} c_{ij}^\alpha e_{ij}^\alpha(g s^{-1}) = \sum_{\alpha \in \mathcal{A}} \sum_{i,j=1}^{n_\alpha} \frac{1}{\sqrt{n_\alpha}} c_{ij}^\alpha \sum_{k=1}^{n_\alpha} e_{ik}^\alpha(g) \overline{e_{jk}^\alpha(s)}. \quad (11)$$

It is easy to verify that if $x = (x_1, \dots, x_{n_\alpha})$ is an eigenvector of $C'_\alpha C_\alpha$ for the eigenvalue λ , then for all $1 \leq k \leq n_\alpha$ the functions $\sum_{j=1}^{n_\alpha} x_j e_{jk}^\alpha(\cdot)$ are eigenfunctions for $T'T$ with eigenvalue $\lambda/\sqrt{n_\alpha}$. Consequently, s_j are the s -numbers of $T'T$. It remains to use Theorem 1.

2. By (11) we have

$$K(g s^{-1}) = \sum_{\alpha \in \mathcal{A}} \frac{1}{\sqrt{n_\alpha}} \sum_{k,i=1}^{n_\alpha} e_{ik}^\alpha(g) \sum_{j=1}^{n_\alpha} c_{ij}^\alpha \overline{e_{jk}^\alpha(s)}.$$

Set

$$E_k^\alpha(g) := \begin{pmatrix} e_{1k}^\alpha(g) \\ \vdots \\ e_{n_\alpha k}^\alpha(g) \end{pmatrix}.$$

Then

$$K(g s^{-1}) = \sum_{\alpha \in \mathcal{A}} \frac{1}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} \left(E_k^\alpha(g), \overline{C}_\alpha E_k^\alpha(s) \right)_\alpha = \sum_{\alpha \in \mathcal{A}} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} \left(E_k^\alpha(g), \overline{U}_\alpha E_k^\alpha(s) \right)_\alpha,$$

where

$$(a, b)_\alpha := \sum_{i=1}^{n_\alpha} a_i \overline{b_i}.$$

There exists an unitary matrix V_α such that

$$V_\alpha U_\alpha V_\alpha^* = \begin{pmatrix} \gamma_1^\alpha & & 0 \\ & \ddots & \\ 0 & & \gamma_{n_\alpha}^\alpha \end{pmatrix} =: \Gamma_\alpha,$$

where $|\gamma_j^\alpha| = 1$, $j = 1, \dots, n_\alpha$. Set

$$F_k^\alpha(g) = \overline{V}_\alpha E_k^\alpha(g) =: \begin{pmatrix} f_{1k}^\alpha(g) \\ \vdots \\ f_{n_\alpha k}^\alpha(g) \end{pmatrix}.$$

Then $E_k^\alpha(g) = \overline{V}_\alpha^* F_k^\alpha(g)$, $f_{ik}^\alpha(g)$ is an orthonormal basis, and

$$\begin{aligned} K(g s^{-1}) &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} \left(\overline{V}_\alpha^* F_k^\alpha(g), \overline{U}_\alpha \overline{V}_\alpha^* F_k^\alpha(s) \right)_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} \left(F_k^\alpha(g), \overline{V}_\alpha \overline{U}_\alpha \overline{V}_\alpha^* F_k^\alpha(s) \right)_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} \left(F_k^\alpha(g), \overline{\Gamma}_\alpha F_k^\alpha(s) \right)_\alpha = \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k,j=1}^{n_\alpha} \gamma_j^\alpha f_{jk}^\alpha(g) \overline{f_{jk}^\alpha(s)}. \end{aligned}$$

Using Theorem 2 we obtain the desired equality. \blacksquare

COROLLARY 1. *Let G be a compact Abelian group, $K(\cdot) \in L_2(G)$, c_j , $j \geq 1$, be Fourier coefficients of K in an orthonormal basis formed by characters of the group. Assume that the c_j are arranged in decreasing order. Then for all $n \in \mathbb{Z}_+$*

$$\begin{aligned} d_n(W_2^K(G), L_2(G)) &= |c_{n+1}|, \\ d_n(W_1^K(G), L_2(G)) &= \left(\sum_{j \geq n+1} |c_j|^2 \right)^{1/2}. \end{aligned}$$

COROLLARY 2. *For $K = (k_1, \dots, k_N)$ set*

$$c_j = \sum_{m=1}^N k_m e^{-2\pi i(j-1)m/N}.$$

Assume that c_j are arranged in decreasing order. Then for all $n \in \mathbb{Z}_+$

$$\begin{aligned} d_n(W_2^K(\mathbb{Z}_N), L_2(\mathbb{Z}_N)) &= |c_{n+1}|, \\ d_n(W_1^K(\mathbb{Z}_N), L_2(\mathbb{Z}_N)) &= \left(\frac{1}{N} \sum_{j \geq n+1} |c_j|^2 \right)^{1/2}. \end{aligned}$$

If $K = (1, 0, \dots, 0)$, then from the last equality we obtain (1).

Let

$$\mathbb{S}^d = \left\{ x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{j=1}^{d+1} x_j^2 = 1 \right\}$$

be the unit sphere. It is known (see [9]) that $L_2(\mathbb{S}^d) = \sum_{k=0}^{\infty} H_k$ where

$$\dim H_k = n_k = \binom{d+k}{k} - \binom{d+k-1}{k-2}$$

(H_k is the set of spherical harmonics of the order k). Let $\{Y_j^k\}_{j=1}^{n_k}$ be an orthonormal basis of H_k . For the Laplace operator Δ and any $x(\cdot) \in H_k$ the equality

$$\Delta x(\cdot) = -\lambda_k x(\cdot)$$

holds where $\lambda_k = k(k + d - 1)$. For $\alpha > 0$ the operator $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}x(\cdot) = \sum_{k=1}^{\infty} \lambda_k^{\alpha/2} \sum_{j=1}^{n_k} x_{kj} Y_j^k(\cdot),$$

where $x(\cdot) \in L_2(\mathbb{S}^d)$ and $x(\cdot) = \sum_{k=0}^{\infty} \sum_{j=1}^{n_k} x_{kj} Y_j^k(\cdot)$.

Set

$$W_2^\alpha(\mathbb{S}^d) = \{x(\cdot) \in L_2(\mathbb{S}^d) \mid \|(-\Delta)^{\alpha/2}x(\cdot)\|_{L_2(\mathbb{S}^d)} \leq 1\}.$$

It is easy to check that this class can be represented in the form

$$W_2^\alpha(\mathbb{S}^d) = \{x(\cdot) \in L_2(\mathbb{S}^d) \mid x(\cdot) = c + Ty(\cdot), \ c \in \mathbb{R}, \ y(\cdot) \in L_2(\mathbb{S}^d), \ y(\cdot) \perp 1\},$$

where for $y(\cdot) = \sum_{k=1}^{\infty} \sum_{j=1}^{n_k} y_{kj} Y_j^k(\cdot)$

$$Ty(\cdot) = \sum_{k=1}^{\infty} \sum_{j=1}^{n_k} \lambda_k^{-\alpha/2} y_{kj} Y_j^k(\cdot).$$

COROLLARY 3. *Let $n_0 + \dots + n_{k-1} \leq n < n_0 + \dots + n_k$. Then*

$$d_n(W_2^\alpha(\mathbb{S}^d), L_2(\mathbb{S}^d)) = \lambda_k^{-\alpha/2}.$$

The class $W_2^\alpha(\mathbb{S}^d)$ for $d = 1$ and $\alpha = r \in \mathbb{Z}_+$ coincides with the Sobolev class

$$W_2^r(\mathbb{T}) = \{x(\cdot) \in L_2(\mathbb{T}) \mid x^{(r-1)}(\cdot) \text{ abs. cont.}, \|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})} \leq 1\}.$$

In this case $\lambda_k = k^2$, $n_0 = 1$, $n_k = 2$, $k \geq 1$. Thus we obtain

COROLLARY 4. *For all $n \in \mathbb{Z}_+$*

$$d_{2n-1}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = d_{2n}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = \frac{1}{n^r}.$$

One does not obtain results similar to those obtained in Corollary 3 and 4 for the classes $W_1^\alpha(\mathbb{S}^d)$ and $W_1^r(\mathbb{T})$. The reason for this is the additional condition $y(\cdot) \perp 1$ which does not permit us to apply Theorem 2. Some estimates of $d_n(W_1^r(\mathbb{T}), L_2(\mathbb{T}))$ may be found in [2, p. 101].

4. AVERAGE WIDTHS

In this section we calculate exact values of average Kolmogorov widths for some classes of functions defined on \mathbb{R}^d and \mathbb{Z}^d in the L_2 metric. We begin with the definition of the average dimension of subspace. Let $G = \mathbb{R}^d$ or \mathbb{Z}^d and μ_G be the Lebesgue measure on G if $G = \mathbb{R}^d$, and discrete measure if $G = \mathbb{Z}^d$. Let $\mathcal{A}(G)$ be the set of positive numbers if $G = \mathbb{R}^d$ and the set of natural numbers

for $G = \mathbb{Z}^d$. Assume that A is a subset of $L_p(G)$ ($1 \leq p \leq \infty$) and $\alpha \in \mathcal{A}(G)$. Denote by A_α the restriction of A to the set

$$\square_\alpha = \{t = (t_1, \dots, t_d) \in G \mid |t_j| \leq \alpha, j = 1, \dots, d\}.$$

Let L be a subspace of $L_p(G)$. For every $\varepsilon > 0$ and $\alpha \in \mathcal{A}(G)$ consider the value

$$K_\varepsilon(\alpha, L, L_p(G)) = \min\{n \in \mathbb{Z}_+ \mid d_n((L \cap BL_p(G))_\alpha, L_p(\square_\alpha)) < \varepsilon\},$$

where $BL_p(G)$ is the unit ball of $L_p(G)$. It is clear that $K_\varepsilon(\alpha, L, L_p(G))$ is the minimal dimension of a subspace of $L_p(\square_\alpha)$ which approximates the set $(L \cap BL_p(G))_\alpha$ to within ε . It is easy to check that for every $\varepsilon > 0$ the function $\alpha \rightarrow K_\varepsilon(\alpha, L, L_p(G))$ does not decrease, and obviously for every $\alpha > 0$ the function $\varepsilon \rightarrow K_\varepsilon(\alpha, L, L_p(G))$ does not increase.

The *average dimension of L in $L_p(G)$* is defined as

$$\overline{\dim}(L, L_p(G)) = \lim_{\varepsilon \rightarrow 0} \liminf_{\alpha \rightarrow \infty} \frac{K_\varepsilon(\alpha, L, L_p(G))}{\mu_G(\square_\alpha)}.$$

It is clear that $\overline{\dim}(L, L_p(\mathbb{Z}^d)) \leq 1$.

We shall formulate here one result about average dimension of functional spaces which we later need. Let G^* be \mathbb{R}^d for $G = \mathbb{R}^d$ and \mathbb{T}^d (a d -dimensional torus) for $G = \mathbb{Z}^d$. Let μ_{G^*} be the Lebesgue measure on G^* divided by $(2\pi)^d$. (The last condition is connected with the fact that μ_G is the natural Haar measure on G as on a locally compact Abelian group and μ_{G^*} is just the Haar measure associated with it on the dual group G^* .)

Let A be a subset of G^* and $1 \leq p \leq \infty$. Set

$$\mathcal{B}_{A,p}(G) = \{x(\cdot) \in L_p(G) \mid \text{supp } \hat{x}(\cdot) \subset A\},$$

where $\text{supp } \hat{x}(\cdot)$ is the support of Fourier transform of $x(\cdot)$ ($x(\cdot)$ is considered as an distribution). It is clear that $\mathcal{B}_{A,p}(G)$ is a subspace of $L_p(G)$.

Recall that a set $A \subset G^*$ is called Jordan measurable if its characteristic function is integrable in the sense of Riemann. The following two theorems (Theorem 4 and 5) were proved in [10, 11] for $G = \mathbb{R}^d$. In a general case these theorems may be proved in much the same way.

THEOREM 4 (ABOUT AVERAGE DIMENSION). *Let $G = \mathbb{R}^d$ or \mathbb{Z}^d , A be a Jordan measurable subset of G^* with finite measure and $1 \leq p \leq \infty$. Then*

$$\overline{\dim}(\mathcal{B}_{A,p}(G), L_p(G)) = \mu_{G^*}(A).$$

The notion of average dimension leads at once to the corresponding analogue of Kolmogorov n -width. Let C be a centrally symmetric subset of $L_p(G)$ and $\nu \geq 0$. The *Kolmogorov average ν -width of C in $L_p(G)$* is defined as follows

$$\overline{d}_\nu(C, L_p(G)) = \inf_L \sup_{x(\cdot) \in C} \inf_{y(\cdot) \in L} \|x(\cdot) - y(\cdot)\|_{L_p(G)},$$

where the first infimum is taken over all subspaces L of $L_p(G)$ such that $\overline{\dim}(L, L_p(G)) \leq \nu$. Any subspace for which this infimum is attained we call an *extremal subspace for $\overline{d}_\nu(C, L_p(G))$* .

The following analogue for average widths of the theorem about widths of a ball holds.

THEOREM 5 (ABOUT WIDTHS OF A BALL). *Let $A \subset G^*$ be a Jordan measurable set, $\nu > 0$, and $\mu_{G^*}(A) > \nu$. then*

$$\overline{d}_\nu(\mathcal{B}_{A,p}(G) \cap BL_p(G), L_p(G)) = 1.$$

We calculate the exact value of the average width in L_2 for the set C which is a convolution class of functions defined on G . If $K(\cdot) \in L_2(G)$, then the operator of convolution with this kernel $x(\cdot) \rightarrow (K * x)(\cdot)$ is obviously a linear continuous operator from $L_2(G)$ into $L_2(G)$. Set

$$W_2^K(G) = \{y(\cdot) \in L_2(G) \mid y(\cdot) = (K * x)(\cdot), \|x(\cdot)\|_{L_2(G)} \leq 1\}.$$

Denote by $\widehat{z}(\cdot)$ the Fourier transform of the function $z(\cdot) \in L_2(G)$.

THEOREM 6. *Let $K(\cdot) \in L_1(G) \cap L_2(G)$, $\nu > 0$ if $G = \mathbb{R}^d$, and $0 < \nu < 1$ if $G = \mathbb{Z}^d$. Then*

$$\overline{d}_\nu(W_2^K(G), L_2(G)) = \widehat{K}^*(\nu),$$

where $\widehat{K}^*(\cdot)$ is the non-decreasing permutation of $\widehat{K}(\cdot)$. Moreover, the space $\mathcal{B}_{A(\nu),2}(G)$, where $A(\nu) = \{\tau \in G^* \mid |\widehat{K}(\tau)| > \widehat{K}^*(\nu)\}$ is an extremal space for $\overline{d}_\nu(W_2^K(G), L_2(G))$.

Proof. The lower bound. We use the method of “embedded balls”. For every $\gamma > 0$ consider the set $A(\gamma) = \{\tau \in G^* \mid |\widehat{K}(\tau)| > \gamma\}$. Since $K(\cdot) \in L_1(G)$ the function $\widehat{K}(\cdot)$ is continuous and $\widehat{K}(\tau) \rightarrow 0$ as $|\tau| \rightarrow \infty$ if $G = \mathbb{R}^d$. Therefore the set $A(\gamma)$ is open and bounded. Thus it is Jordan measurable, as is easy to check. We prove that

$$\mathcal{B}_{A(\gamma),2}(G) \cap \gamma BL_2(G) \subset W_2^K(G). \quad (12)$$

Indeed, let $y(\cdot)$ belong to the left-hand side of (12). Assume that $x(\cdot) \in L_2(G)$ is defined by the condition: $\widehat{x}(\tau) = 0$ almost everywhere if $\tau \notin A(\gamma)$ and $\widehat{x}(\tau) =$

$\widehat{K}^{-1}(\tau)\widehat{y}(\tau)$ almost everywhere if $\tau \in A(\gamma)$. Thus $y(\cdot) = (K * x)(\cdot)$ and we have to show that $\|x(\cdot)\|_{L_2(G)} \leq 1$. By the Plancherel theorem

$$\begin{aligned}\|x(\cdot)\|_{L_2(G)}^2 &= \int_G \left| \widehat{K}^{-1}(\tau)\widehat{y}(\tau) \right|^2 d\tau = \int_{A(\gamma)} \left| \widehat{K}^{-1}(\tau)\widehat{y}(\tau) \right|^2 d\tau \\ &\leq \gamma^{-2} \int_{A(\gamma)} |\widehat{y}(\tau)|^2 d\tau \leq \gamma^{-2} \|y(\cdot)\|_{L_2(G)}^2 \leq 1,\end{aligned}$$

that is, $y(\cdot) \in W_2^K(G)$.

Now $\gamma > 0$ is such that $\mu_{G^*}(A(\gamma)) > \nu$. By Theorem 4

$$\overline{\dim} \left(\mathcal{B}_{A(\gamma),2}(G), L_2(G) \right) = \mu_{G^*}(A(\gamma)) > \nu.$$

Then by Theorem 5 (taking into account the obvious property of homogeneity of these widths) we obtain

$$\overline{d}_\nu \left(\mathcal{B}_{A(\gamma),2}(G) \cap \gamma B L_2(G), L_2(G) \right) = \gamma.$$

From this and (12), using the monotonicity of widths it follows that

$$\overline{d}_\nu \left(W_2^K(G), L_2(G) \right) > \gamma.$$

Passing to the supremum in this inequality over all $\gamma > 0$ for which $\mu_{G^*}(A(\gamma)) > \nu$ (in view of the continuity of $\widehat{K}(\cdot)$ this is equivalent to passage to the supremum over all $\gamma > 0$ for which $\mu_{G^*}(A(\gamma)) \geq \nu$) we obtain the required lower bound.

The upper bound is based on the approximation of $W_2^K(G)$ by the Fourier method. Let $\gamma = \gamma(\nu)$ be such that $\mu_{G^*}(A(\gamma)) = \nu$ (it is clear that in this case $\gamma = \widehat{K}^*(\nu)$). By Theorem 4 $\overline{\dim} \left(\mathcal{B}_{A(\gamma),2}(G), L_2(G) \right) = \nu$. With every $y(\cdot) = (K * x)(\cdot) \in W_2^K(G)$ associate the function $\eta(\cdot) \in \mathcal{B}_{A(\gamma),2}(G)$ such that $\widehat{\eta}(\cdot) = \chi_{A(\gamma)}(\cdot)\widehat{y}(\cdot)$ where $\chi_{A(\gamma)}(\cdot)$ is the characteristic function of the set $A(\gamma)$. ■

The next result is the analogue of (1) for average widths.

THEOREM 7. *Let $0 < \nu < 1$. Then*

$$\overline{d}_\nu \left(B l_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d) \right) = \sqrt{1 - \nu}.$$

Proof. The arguments do not depend on the dimension d so for simplicity we consider the case $d = 1$.

The lower bound. Let L be a subspace of $l_2(\mathbb{Z})$, $\overline{\dim}(L, l_2(\mathbb{Z})) \leq \nu$, and $\varepsilon > 0$. Assume that $\{N_k\}$ is a subsequence of natural numbers such that

$$\liminf_{N \rightarrow \infty} \frac{K_\varepsilon(N, L, l_2(\mathbb{Z}))}{2N + 1} = \lim_{k \rightarrow \infty} \frac{K_\varepsilon(N_k, L, l_2(\mathbb{Z}))}{2N_k + 1}. \quad (13)$$

By the definition of average dimension, for every k there exists a subspace $M_k \subset l_2^{2N_k+1}$ such that

$$d\left((L \cap Bl_2(\mathbb{Z}))_k, M_k, l_2^{2N_k+1}\right) < \varepsilon, \quad (14)$$

$$\dim M_k \leq K_\varepsilon(N_k, L, l_2(\mathbb{Z})). \quad (15)$$

Let $x \in Bl_1^{2N_k+1}$. Extending x by zero to \mathbb{Z} we obtain $x \in Bl_1(\mathbb{Z})$ and consequently $x \in Bl_2(\mathbb{Z})$. Let $y \in L$ and $z \in M_k$ be such that

$$\|y - z\|_{l_2^{2N_k+1}} = d\left(y, M_k, l_2^{2N_k+1}\right). \quad (16)$$

Then using the triangle inequality, (16), (14), and again the triangle inequality, we have

$$\begin{aligned} \|x - y\|_{l_2(\mathbb{Z})} &\geq \|x - y\|_{l_2^{2N_k+1}} \geq \|x - z\|_{l_2^{2N_k+1}} - \|y - z\|_{l_2^{2N_k+1}} \\ &\geq d\left(x, M_k, l_2^{2N_k+1}\right) - d\left(y, M_k, l_2^{2N_k+1}\right) \geq d\left(x, M_k, l_2^{2N_k+1}\right) - \varepsilon\|y\|_{l_2(\mathbb{Z})} \\ &\geq d\left(x, M_k, l_2^{2N_k+1}\right) - \varepsilon\|x - y\|_{l_2(\mathbb{Z})} - \varepsilon\|x\|_{l_2(\mathbb{Z})}. \end{aligned}$$

Consequently,

$$(1 + \varepsilon)\|x - y\|_{l_2(\mathbb{Z})} \geq d\left(x, M_k, l_2^{2N_k+1}\right) - \varepsilon.$$

Hence,

$$(1 + \varepsilon)d\left(Bl_1(\mathbb{Z}), L, l_2(\mathbb{Z})\right) \geq d\left(Bl_1^{2N_k+1}, M_k, l_2^{2N_k+1}\right) - \varepsilon. \quad (17)$$

From (13) (taking into account (15)) it follows that for every $0 < \delta < 1 - \nu$ there exists k_0 such that for all $k \geq k_0$

$$\dim M_k \leq K_\varepsilon(N_k, L, l_2(\mathbb{Z})) \leq (\nu + \delta)(2N_k + 1).$$

Put $N(k) = 2N_k + 1$ and $n(k) = [(\nu + \delta)(2N_k + 1)]$. Then $\dim M_k \leq n(k) < N(k)$. Taking into account these inequalities, (17), and (1) we have

$$\begin{aligned} (1 + \varepsilon)d\left(Bl_1(\mathbb{Z}), L, l_2(\mathbb{Z})\right) &\geq d_{n(k)}\left(Bl_1^{N(k)}, l_2^{N(k)}\right) - \varepsilon \\ &= \sqrt{1 - \frac{n(k)}{N(k)}} - \varepsilon = \sqrt{1 - (\nu + \delta)} - \varepsilon. \end{aligned}$$

In view of the arbitrariness of ε , δ , and L we obtain the required lower bound.

The upper bound. Let $\varepsilon > 0$ and the numbers $n, N \in \mathbb{N}$ be chosen so that $n < N$ and $(n/N) \leq \nu \leq (n/N) + \varepsilon$. Denote by $L_{n,N}$ a subspace of l_2^N with dimension at most n which is extremal for $d_n\left(Bl_1^N, l_2^N\right)$. We consider this subspace as a subspace of functions on \mathbb{Z} with support on $\{0, 1, \dots, N-1\}$. Let $e_i(\cdot)$, $i = 1, \dots, N$, be a basis for $L_{n,N}$. If $k \in \mathbb{Z}$, then the functions $e_i(\cdot + kN)$, $i = 1, \dots, N$, form a basis in the space of all functions from $L_{n,N}$ shifted by kN . Denote by L the set of functions $y(\cdot)$ defined on \mathbb{Z} which have the form

$y(\cdot) = \sum_{k \in \mathbb{Z}} \sum_{i=1}^n x_{ki} e(\cdot + kN)$ where $\sum_{k \in \mathbb{Z}} \sum_{i=1}^n x_{ki}^2 < \infty$. It is clear that L is a subspace of $l_2(\mathbb{Z})$.

We show that $\overline{\dim}(L, l_2(\mathbb{Z})) \leq \nu$. Indeed, denote by L_m the restriction of L to $\{-mN, \dots, mN\}$. It is easy to see that $\dim L_m \leq 2mn + 1$ and therefore

$$\overline{\dim}(L, l_2(\mathbb{Z})) \leq \liminf_{m \rightarrow \infty} \frac{2mn + 1}{2mN + 1} = \frac{n}{N} \leq \nu.$$

Denote by M_k the restriction of L to $\Delta_k = \{kN, \dots, (k+1)N - 1\}$.

Now let $x \in Bl_1(\mathbb{Z})$ and x_k be the restriction of x to Δ_k . Since $x_k \in \|x_k\|_{l_1^N} Bl_1^N$ and $M_k = L_{n,N}$ (if $L_{n,N}$ is considered as a set of functions defined on Δ_k) there exists $y_k \in M_k$ for which

$$\|x_k - y_k\|_{l_2^N} \leq \sqrt{1 - (n/N)} \|x_k\|_{l_1^N}. \quad (18)$$

Let $y \in L$ be a function such that the restriction of y to Δ_k equals y_k . Then using (18) and the mean inequality we have

$$\begin{aligned} \|x - y\|_{l_2(\mathbb{Z})} &= \left(\sum_{k \in \mathbb{Z}} \|x_k - y_k\|_{l_2^N}^2 \right)^{1/2} \leq \sqrt{1 - \frac{n}{N}} \left(\sum_{k \in \mathbb{Z}} \|x_k\|_{l_1^N}^2 \right)^{1/2} \\ &\leq \sqrt{1 - \frac{n}{N}} \sum_{k \in \mathbb{Z}} \|x_k\|_{l_1^N} = \sqrt{1 - \frac{n}{N}} \|x\|_{l_1(\mathbb{Z})} \leq \sqrt{1 - \frac{n}{N}} \leq \sqrt{1 - \nu} + \varepsilon. \end{aligned}$$

In view of the arbitrariness of ε we obtain the required estimate. \blacksquare

We note here one general fact which in particular enables us to obtain at once a series of extremal spaces for the widths $d_n(Bl_1^N, l_2^N)$ and $\overline{d}_\nu(Bl_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d))$.

Let G be a locally compact Abelian group (LCAG), G^* be the dual group to G (that is, the group of all continuous characters on G), and $\text{ch}(g, g^*)$ be the value of $g^* \in G^*$ at the element $g \in G$. We define by μ_G (μ_{G^*}) the Haar measure on G (G^*).

For every $x(\cdot) \in L_1(G)$ the function $\hat{x}(\cdot)$ defined on G^* which is given by the formula

$$\hat{x}(g^*) = \int_G x(g) \text{ch}(-g, g^*) d\mu_G \quad (19)$$

is called the *Fourier transform of $x(\cdot)$* . By (19) it follows that $\hat{x}(\cdot)$ is a continuous function and

$$\|\hat{x}(\cdot)\|_{C(G^*)} \leq \|x(\cdot)\|_{L_1(G)}. \quad (20)$$

The Fourier transform can be extended up to an isometric operator from $L_2(G)$ onto $L_2(G^*)$ (this extension we define by the same symbol $\hat{x}(\cdot)$). Thus we have the Parseval equality

$$\|x(\cdot)\|_{L_2(G)} = \|\hat{x}(\cdot)\|_{L_2(G^*)}. \quad (21)$$

If G is a discrete group, then the dual group G^* is compact and we shall usually assume that $\mu_{G^*}(G^*) = 1$.

Let A be a nonempty subset of G^* and $p = 1$ or 2 . Set

$$\mathcal{B}_{A,p}(G) = \{x(\cdot) \in L_p(G) \mid \text{supp } \hat{x}(\cdot) \subset A\},$$

where $\text{supp } \hat{x}(\cdot)$ is the support of $\hat{x}(\cdot)$. It is clear that $\mathcal{B}_{A,p}(G)$ is a subspace of $L_p(G)$.

PROPOSITION 1. *Let G be a discrete LCAG and A a measurable subset of G^* . Then $L_1(G)$ is embedded in $L_2(G)$ and*

$$d(BL_1(G), \mathcal{B}_{A,2}(G), L_2(G)) \leq \sqrt{1 - \mu_{G^*}(A)}.$$

Proof. Let $x(\cdot) \in BL_1(G)$ and the function $y(\cdot) \in L_2(G)$ be such that $\hat{y}(\cdot) = \chi_A(\cdot)\hat{x}(\cdot)$ ($\chi_A(\cdot)$ is the characteristic function of A). It is clear that $y(\cdot) \in \mathcal{B}_{A,2}(G)$. Using (21) and (20), we have

$$\begin{aligned} \|x(\cdot) - y(\cdot)\|_{L_2(G)}^2 &= \int_{G^* \setminus A} |\hat{x}(g^*)|^2 d\mu_{G^*} \leq \|\hat{x}(\cdot)\|_{C(G^*)}^2 \int_{G^* \setminus A} d\mu_{G^*} \\ &\leq \|x(\cdot)\|_{L_1(G)}^2 (1 - \mu_{G^*}(A)) \leq 1 - \mu_{G^*}(A). \end{aligned}$$

If we take here $x(\cdot) \in L_1(G)$ and $y(\cdot) = 0$, then we obtain that $\|x(\cdot)\|_{L_2(G)} \leq \|x(\cdot)\|_{L_1(G)}$. This means that $L_1(G)$ is continuously embedded in $L_2(G)$. ■

We apply this result to the mentioned above problems.

1. The space l_p^N , $1 \leq p \leq \infty$, can be considered as $L_p(G)$ where $G = \mathbb{Z}_N = \{0, 1, \dots, N-1\}$ is a finite discrete Abelian group with the operation of addition modulo N . Characters of this group are the functions $k \rightarrow \exp(2\pi k l / N)$, $k \in \mathbb{Z}_N$, where $0 \leq l \leq N-1$. Therefore we can identify the dual group \mathbb{Z}_N^* with \mathbb{Z}_N . Let $n < N$ and $A = \{l_{j_1}, \dots, l_{j_n}\} \subset \mathbb{Z}_N^*$. It is clear that $\mu_{\mathbb{Z}_N^*}(A) = n/N$. Consider the space $L_n = \text{span}\{\exp(2\pi i l_{j_1} \cdot / N), \dots, \exp(2\pi i l_{j_n} \cdot / N)\}$, $\dim L_n = n$. From Proposition 1 and (1) it follows that L_n is an extremal subspace for $d_n(Bl_1^N, l_2^N)$.

2. Let $A \subset \mathbb{T}^d$ be Jordan measurable, $\mu_{\mathbb{T}^d}(A) = \nu$, $0 < \nu < 1$. Consider the space $L_\nu = \{x(\cdot) \in l_2(\mathbb{Z}^d) \mid \text{supp } \hat{x}(\cdot) \subset A\}$. By Theorem 4 we have $\overline{\dim}(L_\nu, l_2(\mathbb{Z}^d)) = \nu$. Now from Proposition 1 and Theorem 4 it follows that L_ν is an extremal subspace for $\overline{d}_\nu(Bl_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d))$.

5. COMMENTS

Various statements which are equivalent to Theorem 1 were proved by many authors (see [12, 13, 14, 15, 2]). Of course this result was known to Kolmogorov who considered in [3] only particular cases of elliptical cylinders.

In a finite-dimensional space n -widths of regular octahedra were in fact obtained in two papers [4] (the upper bound) and [5] (the lower bound). It is interesting to note that Kolmogorov in 1948 did not take into consideration that in these papers $d_n(Bl_1^N, l_2^N)$ were calculated. This fact was noted by Stechkin [6] who used it to find asymptotic values of n -widths for functional classes.

There is one more type of octahedra for which it is possible to calculate exact values of widths. They are octahedra with different axes

$$Bl_1^N(a) := \left\{ x \in \mathbb{R}^N \mid \sum_{k=1}^N \frac{|x_k|}{a_k} \leq 1 \right\}, \quad a_1 \geq \dots \geq a_N.$$

For the dual case Smolyak [16] found the exact values of the linear (λ_n) and Gel'fand (d^n) n -widths

$$\lambda_n(Bl_2^N(a), l_\infty^N) = d^n(Bl_2^N(a), l_\infty^N) = \max_{m \geq n} \sqrt{\frac{m-n}{\sum_{k=1}^m a_k^{-2}}}.$$

For the Kolmogorov n -width $d_n(Bl_1(a), l_2)$ the exact result was obtained by Sofman [17, 18] (see also [19]).

In the continuous case estimates for the n -widths of generalized octahedra and even more general sets (images of compacts under continuous transformation in the Hilbert space) can be obtained using results such as a theorem of Ismagilov [20] which is based on the method of averages (we demonstrated this method in the proof of Theorem 2). Ismagilov cited Obukhov [21] as a predecessor in using the method of averages. Several statements of a similar type which are used for calculating exact values of n -widths for classes of analytic functions can be found in [22, 23, 24]. In those papers the dual situation is considered and the exact values of linear, Gel'fand, and Bernstein widths of $W_2^K(X)$ in $C(X)$ are found. In the dual case using the Hilbert space structure it is possible to calculate the exact values of n -widths for $W_2^\alpha(\mathbb{S}^d)$ and $W_2^r(\mathbb{T})$.

The concept of average dimension takes the beginning from the definition of “average entropy” for stochastic signals with bounded spectrum which was offered by Shannon [25]. Further Kolmogorov modified this definition for determined functions. Then Tikhomirov [26] defined the notion of average dimension replacing entropy by Kolmogorov widths. The definition of the average dimension used in the paper is a modification of Tikhomirov's definition. The notion of the Kolmogorov average widths is due to Magaril-Il'yaev.

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