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RESEARCH ARTICLE

Extremal Problems for the Generalized Heat Equation and Optimal Recovery of its Solution from Inaccurate Data

K. Yu. Osipenko

"MATI" — Russian State Technological University
121552 Orshanskaya 3, Moscow, Russia. Email: kosipenko@yahoo.com
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We consider some extremal problems for the solution of the generalized heat equation similar to the well-known Hadamard three-circle theorem. These problems are connected with optimal recovery problems. We construct a family of optimal recovery methods for the solution of the generalized heat equation using the information about inaccurate data.

Keywords: extremal problems, optimal recovery, heat equation

AMS Subject Classification: 41A65; 35K05; 46E20; 46E35

1. Introduction

We begin with one extremal problem which is known as the Hadamard three-circle theorem (see, for example, [8]). Let $f(z)$ be a holomorphic function on the annulus

$$r_1 \le |z| \le r_2.$$

Put

$$M(r) = \max_{|z|=r} |f(z)|.$$

Then $\log M(r)$ is a convex function of the $\log r$. The conclusion of the theorem can be restated as

$$M(r) \le M(r_1)^{\frac{\log r_2/r}{\log r_2/r_1}} M(r_2)^{\frac{\log r/r_1}{\log r_2/r_1}}.$$

for any three concentric circles of radii $r_1 < r < r_2$.

The history of this theorem is the following. A statement and proof for the theorem was given by J.E. Littlewood in 1912, but he attributes it to no one in particular, stating it as a known theorem. H. Bohr and E. Landau claim the theorem was first given by J. Hadamard in 1896, although Hadamard had published no proof.

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The Hadamard three-circle theorem gives an estimate in the following extremal problem

$$M(r) \to \max, \quad M(r_1) \le \delta_1, \quad M(r_2) \le \delta_2.$$

The exact solution of this problem which is expressed in terms of elliptic functions was given by R. M. Robinson in 1943 [6].

In 1913 E. Landau [3] considered a very similar problem. He took derivatives instead of circles. He proved that for all functions $x \in L_\infty(\mathbb{R}_+)$ with the first derivative locally absolutely continuous on $\mathbb{R}_+$ and $x'' \in L_\infty(\mathbb{R}_+)$ the following exact inequality

$$\|x'\|_{L_\infty(\mathbb{R}_+)} \le 2\|x\|^{1/2}_{L_\infty(\mathbb{R}_+)}\|x''\|^{1/2}_{L_\infty(\mathbb{R}_+)}$$

holds (the exactness means that the constant 2 could not be replaced by some other constant which is less than 2). That is he found the exact solution of the extremal problem

$$\|x'\|_{L_\infty(\mathbb{R}_+)} \to \max, \quad \|x\|_{L_\infty(\mathbb{R}_+)} \le \delta_1, \quad \|x''\|_{L_\infty(\mathbb{R}_+)} \le \delta_2.$$

Then in 1914 Hadamard [1] solved the analogous problem for $\mathbb{R}$.

In 1939 A. N. Kolmogorov [2] obtained the general result in this field. He found the exact solution of the problem

$$\|x^{(k)}\|_{L_\infty(\mathbb{R})} \to \max, \quad \|x\|_{L_\infty(\mathbb{R})} \le \delta_1, \quad \|x^{(r)}\|_{L_\infty(\mathbb{R})} \le \delta_2.$$

The value of this problem is

$$\frac{K_{r-k}}{K_r^{1-\frac{k}{r}}}\delta_1^{1-k/r}\delta_2^{k/r},$$

where

$$K_m = \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^{s(m+1)}}{(2s+1)^{m+1}}$$

are the Favard constants.

These types of extremal problems are known as Landau–Kolmogorov inequalities for derivatives and they all are similar to the initial extremal problem formulated by Hadamard.

2. Analog of the Hadamard theorem for the heat equation

We will consider the problem which is analogous to the Hadamard three-circle theorem but the role of circles will play the time.

Let u be the solution of the generalized heat equation in $\mathbb{R}^d$

$$\begin{aligned} &u_t + (-\Delta)^{\alpha/2}u = 0, \\ &u_{|t=0} = f(x), \quad f \in L_2(\mathbb{R}^d). \end{aligned} \tag{1}$$

The operator $(-\Delta)^{\alpha/2}$ is defined as

$$(-\Delta)^{\alpha/2}g(x) = F^{-1}(|\xi|^\alpha Fg(\xi))(x),$$

where F is the Fourier transform in $L_2(\mathbb{R}^d)$ and F^{-1} is the inverse Fourier transform, and the boundary condition means that $\|u(t,x) - f(x)\|_{L_2(\mathbb{R}^d)} \to 0$ as $t \to 0$.

It is easy to show that the unique solution of (1) is the function

$$u(t,x) = F^{-1}(e^{-|\xi|^\alpha t} Ff(\xi))(x), \tag{2}$$

Theorem 2.1: *For any solution of the generalized heat equation* (1) $\log \|u(t,\cdot)\|_{L_2(\mathbb{R}^d)}$ *is a convex function of* t.

Proof: By Plancherel's theorem it follows from (2) that

$$\|u(t,x)\|^2_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-2|\xi|^\alpha t} |Ff(\xi)|^2 \, d\xi.$$

Let $0 \le t_1 < \tau < t_2$. Put

$$p = \frac{t_2 - t_1}{t_2 - \tau}, \quad q = \frac{t_2 - t_1}{\tau - t_1}, \quad d\mu(\xi) = \frac{1}{(2\pi)^d} |Ff(\xi)|^2 \, d\xi.$$

Then using the fact that $\tau = t_1/p + t_2/q$ and Hölder's inequality we obtain

$$\begin{aligned} \|u(\tau,x)\|^2_{L_2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} e^{-2|\xi|^\alpha (t_1/p + t_2/q)} \, d\mu(\xi) \\ &\le \left(\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha t_1} \, d\mu(\xi) \right)^{1/p} \left(\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha t_2} \, d\mu(\xi) \right)^{1/q} \\ &= \|u(t_1,x)\|^{2\frac{t_2-\tau}{t_2-t_1}}_{L_2(\mathbb{R}^d)} \|u(t_2,x)\|^{2\frac{\tau-t_1}{t_2-t_1}}_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Thus we proved that

$$\|u(\tau,x)\|_{L_2(\mathbb{R}^d)} \le \|u(t_1,x)\|^{\frac{t_2-\tau}{t_2-t_1}}_{L_2(\mathbb{R}^d)} \|u(t_2,x)\|^{\frac{\tau-t_1}{t_2-t_1}}_{L_2(\mathbb{R}^d)}.$$

It means that $\log \|u(t,\cdot)\|_{L_2(\mathbb{R}^d)}$ is a convex function of t. □

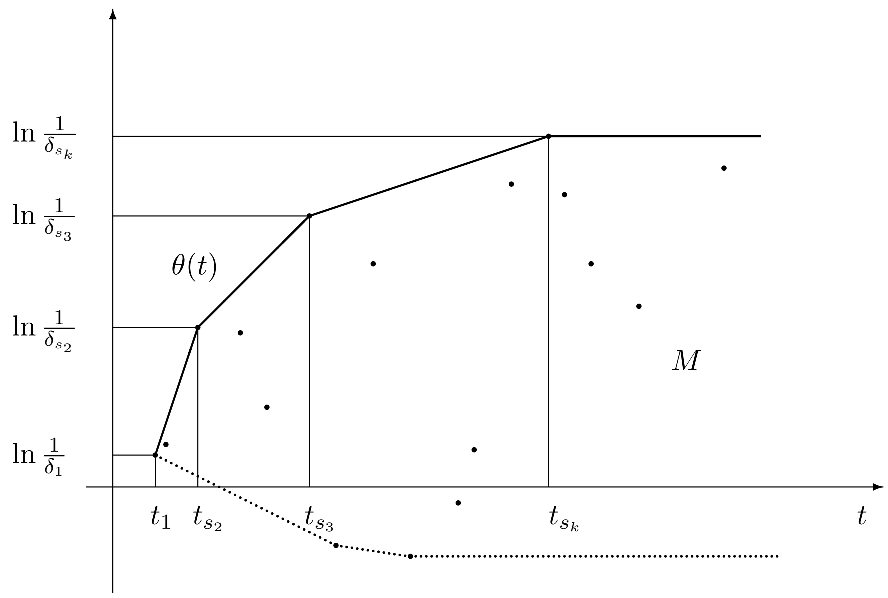
Now we consider the similar problem with $n+1$ "circles". Namely, we want to solve the following extremal problem

$$\|u(\tau,x)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \|u(t_j,x)\|_{L_2(\mathbb{R}^d)} \le \delta_j, \quad j = 1,2,\dots,n, \qquad f \in L_2(\mathbb{R}^d),$$

where $0 \le t_1 < \ldots < t_n$ and $\delta_j > 0$, $j = 1,2,\dots,n$.

To formulate the result we consider the set

$$M = \mathrm{co}\{\, (t_j, \ln(1/\delta_j)),\ 1 \le j \le n \,\} + \{\, (t,0) \mid\ t \ge 0 \,\},$$



where co A is a convex hull of A. Define the function $\theta(t)$, $t \in [t_1, \infty)$ as follows

$$\theta(t) = \max\{\, s \mid (t, s) \in M \,\}.$$

It is clear that θ is a polygonal line on $[t_1, \infty)$.

Theorem 2.2: *For all $\tau \geq t_1$*

$$\sup_{\substack{f \in L_2(\mathbb{R}^d) \\ \|u(t_j,x)\|_{L_2(\mathbb{R}^d)} \leq \delta_j,\ j=1,2,\dots,n}} \|u(\tau, x)\|_{L_2(\mathbb{R}^d)} = e^{-\theta(\tau)}.$$

The proof of this theorem may be obtained by the same scheme as in [4] (where the case $\alpha = 2$ was considered).

3. Optimal recovery of the solution of the heat equation

This extremal problem is closely connected with the problem of optimal recovery of the solution of the heat equation at the instant of time τ knowing inaccurate observations of the solution at the instants $t_1, \dots, t_n$. Let us formulate this recovery problem more precisely.

Assume that we know functions $y_j \in L_2(\mathbb{R}^d)$, $j = 1, \dots, n$, such that

$$\|u(t_j, x) - y_j(x)\|_{L_2(\mathbb{R}^d)} \leq \delta_j, \quad j = 1, \dots, n.$$

What is the best way to use this information to recover the temperature distribution at the time $\tau \neq t_j$, $1 \leq j \leq n$, that is to recover the function $u(\tau, x)$?

We admit as recovery methods arbitrary maps $m\colon (L_2(\mathbb{R}^d))^n \to L_2(\mathbb{R}^d)$. For a fixed method m the quantity

$$e_\tau(L_2(\mathbb{R}^d),\delta,m) = \sup_{\substack{f,y_1,\dots,y_n\in L_2(\mathbb{R}^d)\\ \|u(t_j,x)-y_j(x)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\ j=1,\dots,n}} \|u(\tau,x)-m(y)(x)\|_{L_2(\mathbb{R}^d)},$$

where u is the solution of (1), $\delta=(\delta_1,\dots,\delta_n)$, and $y=(y_1,\dots,y_n)$, is called the error of the method m.

We are interested in the value

$$E_\tau(L_2(\mathbb{R}^d),\delta) = \inf_{m\colon (L_2(\mathbb{R}^d))^n\to L_2(\mathbb{R}^d)} e_\tau(L_2(\mathbb{R}^d),\delta,m),$$

which is called the error of optimal recovery and in the method $\widehat{m}$, for which the infinum is attained that is in the method $\widehat{m}$ for which

$$E_\tau(L_2(\mathbb{R}^d),\delta) = e_\tau(L_2(\mathbb{R}^d),\delta,\widehat{m}).$$

We call this method the optimal recovery method.

Theorem 3.1 : *For all $\tau\ge t_1$*

$$E_\tau(L_2(\mathbb{R}^d),\delta) = e^{-\theta(\tau)}.$$

If $t_{s_1}<\ldots<t_{s_k}$ are the points of break of polygonal line θ and $t_{s_j}<\tau<t_{s_{j+1}}$, then the method

$$\widehat{m}(y) = K_{s_j} * y_{s_j} + K_{s_{j+1}} * y_{s_{j+1}}$$

is optimal; here

$$FK_{s_j}(\xi) = \frac{(t_{s_{j+1}}-\tau)\delta_{s_{j+1}}^2 e^{|\xi|^\alpha(t_{s_{j+1}}-\tau)}}{(t_{s_{j+1}}-\tau)\delta_{s_{j+1}}^2 e^{|\xi|^\alpha(t_{s_{j+1}}-t_{s_j})} + (\tau-t_{s_j})\delta_{s_j}^2 e^{-|\xi|^\alpha(t_{s_{j+1}}-t_{s_j})}},$$

$$FK_{s_{j+1}}(\xi) = \frac{(\tau-t_{s_j})\delta_{s_j}^2 e^{-|\xi|^\alpha(\tau-t_{s_j})}}{(t_{s_{j+1}}-\tau)\delta_{s_{j+1}}^2 e^{|\xi|^\alpha(t_{s_{j+1}}-t_{s_j})} + (\tau-t_{s_j})\delta_{s_j}^2 e^{-|\xi|^\alpha(t_{s_{j+1}}-t_{s_j})}}.$$

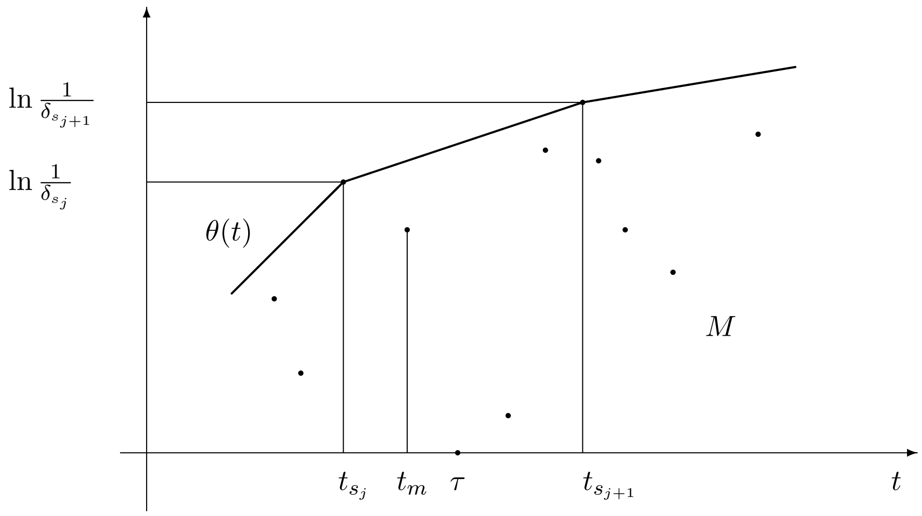
If $\tau>t_{s_k}$, then the method which is solution of the heat equation with the initial condition

$$u\big|_{t=t_{s_k}} = y_{s_k}(x)$$

at the instant of time τ is optimal.

The proof of this theorem for $\alpha=2$ may be found in [4].

Note that the optimal method of recovery $\widehat{m}$ uses not more than two observations. To find these observations we have to construct the set M and the polygonal line θ. Then we have to find the nearest points of break of θ to the point τ. The



observations at these points are those that are used in the optimal method of recovery $\widehat{m}$.

Note also that we can make more precise points of observation which are not on the polygonal line. Suppose that for some t_m, $t_{s_j} < t_m < t_{s_{j+1}}$ and

$$\log \frac{1}{\delta_m} < \theta(t_m).$$

Then optimal recovery method gives the error less than δ_m. Indeed

$$\|u(t_m, x) - \widehat{m}(y)(x)\|_{L_2(\mathbb{R}^d)} \le e^{-\theta(t_m)} < \delta_m.$$

Further investigations discovered a surprising fact. It was found that parallel with the optimal method from the previous theorem there are a lot of optimal recovery methods. To discuss this situation we consider the recovery problem for two instant of times t_1 and t_2. Without loss of generality we assume that $t_1 = 0$ and $t_2 = T$.

The scheme of obtaining of the optimal recovery method $\widehat{m}$ is the following. First, we consider the extremal problem

$$\|u(\tau, x)\|^2_{L_2(\mathbb{R}^d)} \to \max, \quad \|u(0, x)\|^2_{L_2(\mathbb{R}^d)} = \|f\|^2_{L_2(\mathbb{R}^d)} \le \delta_0^2,$$
$$\|u(T, x)\|^2_{L_2(\mathbb{R}^d)} \le \delta_T^2, \quad f \in L_2(\mathbb{R}^d).$$

Passing to the Fourier transform and using the Plancherel theorem we obtain the following problem

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-2|\xi|^\alpha \tau} |Ff(\xi)|^2 \, d\xi \to \max, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |Ff(\xi)|^2 \, d\xi \le \delta_0^2,$$
$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-2|\xi|^\alpha T} |Ff(\xi)|^2 \, d\xi \le \delta_T^2, \quad f \in L_2(\mathbb{R}^d).$$

There is no existence in this problem. We consider the following extension of this problem (changing $(2\pi)^{-d}|Ff(\xi)|^2\,d\xi$ by a positive measure $d\mu(\xi)$):

$$\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha\tau}\,d\mu(\xi)\to\max,\quad \int_{\mathbb{R}^d} d\mu(\xi)\le\delta_0^2,$$
$$\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha T}\,d\mu(\xi)\le\delta_T^2,\quad d\mu(\xi)\ge 0. \tag{3}$$

To solve this extremal problem we consider the Lagrange function of this problem

$$\mathcal{L}(d\mu,\lambda_0,\lambda_T)=\int_{\mathbb{R}^d}\left(-e^{-2|\xi|^\alpha\tau}+\lambda_0+\lambda_T e^{-2|\xi|^\alpha T}\right)d\mu(\xi).$$

Then we find an extremal measure $d\widehat{\mu}(\xi)$ and the Lagrange multipliers $\widehat{\lambda}_0,\widehat{\lambda}_T$ such that

$$(a)\quad \min_{d\mu(\xi)\ge 0}\mathcal{L}(d\mu,\widehat{\lambda}_0,\widehat{\lambda}_T)=\mathcal{L}(d\widehat{\mu},\widehat{\lambda}_0,\widehat{\lambda}_T),$$
$$(b)\quad \widehat{\lambda}_0\left(\int_{\mathbb{R}^d} d\widehat{\mu}(\xi)-\delta_0^2\right)+\widehat{\lambda}_T\left(\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha T}\,d\widehat{\mu}(\xi)-\delta_T^2\right)=0.$$

Then for arbitrary $y_0,y_T\in L_2(\mathbb{R}^d)$ we consider the extremal problem

$$\widehat{\lambda}_0\|f(x)-y_0(x)\|^2_{L_2(\mathbb{R}^d)}+\widehat{\lambda}_T\|u(T,x)-y_T(x)\|^2_{L_2(\mathbb{R}^d)}\to\min,\quad f\in L_2(\mathbb{R}^d),$$

where u is the solution of the generalized heat equation (1) with the initial temperature distribution f. If $\widehat{f}$ is the solution of this problem, then the method

$$\widehat{m}(y_0,y_T)(x)=\widehat{u}(\tau,x),$$

where $\widehat{u}$ is the solution of (1) with the initial temperature distribution $\widehat{f}$, is an optimal method of recovery (see [4] for details).

Let us consider more explicitly extremal problem (3). For any $\sigma_0,\sigma_T>0$ the value of the extended problem

$$\int_{\mathbb{R}^d} e^{-2|\xi|^\alpha\tau}\,d\mu(\xi)\to\max,\quad \int_{|\xi|\ge\sigma_0} d\mu(\xi)\le\delta_0^2,$$
$$\int_{|\xi|\le\sigma_T} e^{-2|\xi|^\alpha T}\,d\mu(\xi)\le\delta_T^2,\quad d\mu(\xi)\ge 0 \tag{4}$$

is not less than the value of (3).

The reason of the existence of a collection of optimal recovery methods is in the fact that there is a set of $\sigma_0,\sigma_T>0$ for which the values of these extremal problems are the same.

Assume that $\delta_T < \delta_0$. Set

$$\widehat{\sigma}_0 = \begin{cases} \left(\dfrac{1}{2T}\log\left(\left(\dfrac{\tau}{T}\right)^{\frac{T}{T-\tau}}\dfrac{\delta_0^2}{\delta_T^2}\right)\right)^{1/\alpha}, & \dfrac{\delta_T^2}{\delta_0^2} < \left(\dfrac{\tau}{T}\right)^{\frac{T}{T-\tau}}, \\ 0, & \dfrac{\delta_T^2}{\delta_0^2} \geq \left(\dfrac{\tau}{T}\right)^{\frac{T}{T-\tau}}, \end{cases}$$

$$\widehat{\sigma}_T = \left(\frac{1}{2T}\log\left(\left(\frac{T}{T-\tau}\right)^{\frac{T}{\tau}}\frac{\delta_0^2}{\delta_T^2}\right)\right)^{1/\alpha}.$$

Theorem 3.2: *If $\delta_T < \delta_0$, then for all $0 \leq \sigma_0 \leq \widehat{\sigma}_0$ and $\sigma_T \geq \widehat{\sigma}_T$ the values of problems (3) and (4) are the same.*

Proof: Put $d\widehat{\mu}(\xi) = \delta_0^2\delta(\xi - \xi_0)$, where ξ_0 such that

$$|\xi_0|^\alpha = \frac{1}{T}\log\frac{\delta_0}{\delta_T}$$

and $\delta(\xi - \xi_0)$ is the delta-function at the point ξ_0. Then it is easy to verify that condition (b) is fulfilled.

The Lagrange function of problem (3) may be written in the form

$$\mathcal{L}(d\mu, \lambda_0, \lambda_T) = \int_{\mathbb{R}^d} e^{-2|\xi|^\alpha \tau} f(|\xi|^\alpha)\, d\mu(\xi),$$

where

$$g(v) = -1 + \lambda_0 e^{2v\tau} + \lambda_T e^{-2v(T-\tau)}.$$

The function g is convex. So if $g(v_0) = g'(v_0) = 0$, then $g(v) \geq 0$ for all v. Put $v_0 = |\xi_0|^\alpha$ and chose $\widehat{\lambda}_0, \widehat{\lambda}_T$ from the condition $g(v_0) = g'(v_0) = 0$. It is easy to obtain that

$$\widehat{\lambda}_0 = \frac{T-\tau}{T}\left(\frac{\delta_T}{\delta_0}\right)^{\frac{2\tau}{T}}, \quad \widehat{\lambda}_T = \frac{\tau}{T}\left(\frac{\delta_0}{\delta_T}\right)^{\frac{2(T-\tau)}{T}}.$$

Consequently, we have

$$-e^{-2|\xi|^\alpha\tau} + \lambda_0 + \lambda_T e^{-2|\xi|^\alpha T} \geq 0$$

for all $\xi \in \mathbb{R}^d$. Thus $\mathcal{L}(d\mu, \widehat{\lambda}_0, \widehat{\lambda}_T) \geq 0$ for all $d\mu(\xi) \geq 0$ and $\mathcal{L}(d\widehat{\mu}, \widehat{\lambda}_0, \widehat{\lambda}_T) = 0$. It means that condition (a) is also fulfilled and $d\widehat{\mu}(\xi)$ is the extremal measure in problem (3).

Consider the Lagrange function of problem (4)

$$\widetilde{\mathcal{L}}(d\mu, \lambda_0, \lambda_T) = \int_{\mathbb{R}^d} -e^{-2|\xi|^\alpha\tau}\, d\mu(\xi) + \lambda_0 \int_{|\xi|\geq\sigma_0} d\mu(\xi) \\ + \lambda_T \int_{|\xi|\leq\sigma_T} e^{-2|\xi|^\alpha T}\, d\mu(\xi).$$

It remains to show that for the same $\widehat{\lambda}_0, \widehat{\lambda}_T$ as above conditions

$$(a') \qquad \min_{d\mu(\xi)\geq 0} \widetilde{\mathcal{L}}(d\mu, \widehat{\lambda}_0, \widehat{\lambda}_T) = \widetilde{\mathcal{L}}(d\widehat{\mu}, \widehat{\lambda}_0, \widehat{\lambda}_T),$$

$$(b') \quad \widehat{\lambda}_0 \left(\int_{|\xi|\geq\sigma_0} d\widehat{\mu}(\xi) - \delta_0^2 \right) + \widehat{\lambda}_T \left(\int_{|\xi|\leq\sigma_T} e^{-2|\xi|^\alpha T}\, d\widehat{\mu}(\xi) - \delta_T^2 \right) = 0$$

are fulfilled. Indeed,

$$\widetilde{\mathcal{L}}(d\mu, \widehat{\lambda}_0, \widehat{\lambda}_T) = \int_{|\xi|<\sigma_0} (-e^{-2|\xi|^\alpha\tau} + \lambda_T e^{-2|\xi|^\alpha T})\, d\mu(\xi)$$
$$+ \int_{\sigma_0\leq|\xi|\leq\sigma_T} e^{-2|\xi|^\alpha\tau} f(|\xi|^\alpha)\, d\mu(\xi) + \int_{|\xi|>\sigma_T} (-e^{-2|\xi|^\alpha\tau} + \widehat{\lambda}_0)\, d\mu(\xi).$$

If $\widehat{\sigma}_0 > 0$, then for all $\sigma_0 \leq \widehat{\sigma}_0$ and all ξ such that $|\xi| < \sigma_0$

$$-e^{-2|\xi|^\alpha\tau} + \lambda_T e^{-2|\xi|^\alpha T} \geq 0.$$

Moreover, for all $\sigma_T \geq \widehat{\sigma}_T$ and all ξ such that $|\xi| > \sigma_T$

$$-e^{-2|\xi|^\alpha\tau} + \widehat{\lambda}_0 \geq 0.$$

Thus $\widetilde{\mathcal{L}}(d\mu, \widehat{\lambda}_0, \widehat{\lambda}_T) \geq 0$ for all $d\mu(\xi) \geq 0$. Since $\widetilde{\mathcal{L}}(d\widehat{\mu}, \widehat{\lambda}_0, \widehat{\lambda}_T) = 0$ condition (a') holds. It is easy to check that condition (b') is fulfilled. Consequently, $d\widehat{\mu}(\xi)$ is extremal measure for problem (4). $\square$

Using Theorem 3.2 we obtain the following result

Theorem 3.3: *For all $0 \leq \sigma_0 \leq \widehat{\sigma}_0$ and $\sigma_T \geq \widehat{\sigma}_T$ the methods*

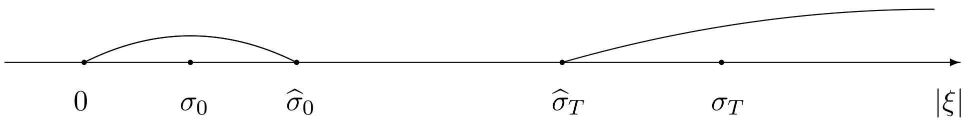
$$\widehat{m}_{\sigma_0,\sigma_T}(y_0, y_T) = K_0 * y_0 + K_T * y_T$$

are optimal; here

$$FK_0(\xi) = \begin{cases} 0, & 0 \leq |\xi| \leq \sigma_0, \\ \dfrac{(T-\tau)\delta_T^2 e^{|\xi|^\alpha(T-\tau)}}{(T-\tau)\delta_T^2 e^{|\xi|^\alpha T} + \tau\delta_0^2 e^{-|\xi|^\alpha T}}, & \sigma_0 < |\xi| < \sigma_T, \\ e^{-|\xi|^\alpha\tau}, & |\xi| \geq \sigma_T, \end{cases}$$

$$FK_T(\xi) = \begin{cases} e^{|\xi|^\alpha(T-\tau)}, & 0 \leq |\xi| \leq \sigma_0, \\ \dfrac{\tau\delta_0^2 e^{-|\xi|^\alpha\tau}}{(T-\tau)\delta_T^2 e^{|\xi|^\alpha T} + \tau\delta_0^2 e^{-|\xi|^\alpha T}}, & \sigma_0 < |\xi| < \sigma_T, \\ 0, & |\xi| \geq \sigma_T, \end{cases}$$

Theorem 3.3 gives a family of optimal recovery methods depending on two parameters σ_0 and σ_T.



4. Optimal recovery for the heat equation in the d-dimensional ball

Now we consider the analogous problems for the unit d-dimensional ball.

Set

$$\mathbb{B}^d = \left\{ x = (x_1, \dots, x_d) : |x|^2 = \sum_{j=1}^d x_j^2 < 1 \right\},$$

$$\mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}.$$

Let u be the solution of the generalized heat equation in $\mathbb{B}^d$:

$$\begin{aligned} &u_t + (-\Delta)^{\alpha/2} u = 0, \quad \alpha > 0, \\ &u|_{t=0} = f(x), \quad f \in L_2(\mathbb{B}^d) \\ &u|_{x \in \mathbb{S}^{d-1}} = 0. \end{aligned} \tag{5}$$

We would like to consider the following extremal problem

$$\|u(\tau, x)\|_{L_2(\mathbb{B}^d)} \to \max, \quad \|u(t_j, x)\|_{L_2(\mathbb{B}^d)} \le \delta_j, \ j = 1, 2, \qquad f \in L_2(\mathbb{B}^d), \tag{6}$$

where u is the solution of problem (5).

This extremal problem is closely connected with the problem of optimal recovery of the solution at the instant of time τ knowing inaccurate observations of the solution at the instants t_1 and t_2. Let us formulate this recovery problem more precisely.

Suppose that we know two functions $y_1, y_2 \in L_2(\mathbb{B}^d)$ such that

$$\|u(t_j, x) - y_j(x)\|_{L_2(\mathbb{B}^d)} \le \delta_j, \quad j = 1, 2.$$

We want to recover the solution at the instant τ using inaccurate observations y_1 and y_2.

We admit as recovery methods any maps $m\colon L_2(\mathbb{B}^d) \times L_2(\mathbb{B}^d) \to L_2(\mathbb{B}^d)$. For a fixed method m the quantity

$$e_\tau(L_2(\mathbb{B}^d), \delta_1, \delta_2, m) = \sup_{\substack{f, y_1, y_2 \in L_2(\mathbb{B}^d) \\ \|u(t_j, x) - y_j(x)\|_{L_2(\mathbb{B}^d)} \le \delta_j, \ j=1,2}} \|u(\tau, x) - m(y_1, y_2)(x)\|_{L_2(\mathbb{B}^d)},$$

where u is the solution of (5), is called the error of the method m.

The quantity

$$E_\tau(L_2(\mathbb{B}^d),\delta_1,\delta_2)=\inf_{m\colon L_2(\mathbb{B}^d)\times L_2(\mathbb{B}^d)\to L_2(\mathbb{B}^d)} e_\tau(L_2(\mathbb{B}^d),\delta_1,\delta_2,m)$$

is called the error of optimal recovery and a method delivering the lower bound is called an optimal recovery method.

It appears that the error of optimal recovery equals the value of extremal problem (6). That is,

$$E_\tau(L_2(\mathbb{B}^d),\delta_1,\delta_2)=\sup_{\substack{f\in L_2(\mathbb{B}^d)\\ \|u(t_j,x)\|_{L_2(\mathbb{B}^d)}\le\delta_j,\ j=1,2}}\|u(\tau,x)\|_{L_2(\mathbb{B}^d)}.$$

To formulate the result recall some facts about the eigenfunctions of the Laplace operator. We consider the case $d>1$. Let H_k denote the set of spherical harmonics of order k. It is known (see [7]) that $\dim H_0=a_0=1$,

$$\dim H_k=a_k=(d+2k-2)\frac{(d+k-3)!}{(d-2)!k!},\quad k=1,2,\dots,$$

and

$$L_2(\mathbb{S}^{d-1})=\sum_{k=0}^\infty H_k.$$

Let $\{Y_j^{(k)}\}_{j=1}^{a_k}$ denote an orthonormal basis in H_k. Let J_p be the Bessel function of the first kind of order p and $\mu_s^{(p)}$, $s=1,2,\dots$, be the zeros of J_p.

The functions

$$Z_{skj}(x)=\frac{J_p(\mu_s^{(p)}r)}{r^{d/2-1}}Y_j^{(k)}(x'),$$

where $r=|x|$, $x'=x/r$, and $p=k+(d-2)/2$, form an orthogonal basis in $L_2(\mathbb{B}^d)$. Moreover,

$$\Delta Z_{skj}=-(\mu_s^{(p)})^2Z_{skj}.$$

We will use the orthonormal basis in $L_2(\mathbb{B}^d)$

$$Y_{skj}=\frac{Z_{skj}}{\|Z_{skj}\|_{L_2(\mathbb{B}^d)}}.$$

We recall that the operator $(-\Delta)^{\alpha/2}$ is defined as follows

$$(-\Delta)^{\alpha/2}f=\sum_{s=1}^\infty\sum_{k=0}^\infty(\mu_s^{(p)})^\alpha\sum_{j=1}^{a_k}c_{skj}Y_{skj},$$

where

$$f=\sum_{s=1}^\infty\sum_{k=0}^\infty\sum_{j=1}^{a_k}c_{skj}Y_{skj}.\tag{7}$$

The solution of (5) can be easily found by the Fourier method of separation of variables. It has the form

$$u(t,x)=\sum_{s=1}^{\infty}\sum_{k=0}^{\infty}e^{-(\mu_s^{(p)})^\alpha t}\sum_{j=1}^{a_k}c_{skj}Y_{skj}(x),$$

where c_{skj} are the Fourier coefficients of the initial function.

Set

$$a_{sk}=e^{-2(\mu_s^{(p)})^\alpha}.$$

The zeros of the Bessel functions $\mu_s^{(p)}$, $s=1,2,\ldots$, $p=k+(d-2)/2$, $k=0,1,\ldots$, can be arranged in ascending order

$$\mu_{s_1}^{(p_1)}<\mu_{s_2}^{(p_2)}<\ldots<\mu_{s_n}^{(p_n)}<\ldots.$$

Put $\beta_m=a_{s_mk_m}$. Then

$$\beta_1>\beta_2>\ldots>\beta_n>\ldots.$$

We introduce the following notation

$$\Delta_m=\left[\beta_{m+1}^{t_2-t_1},\beta_m^{t_2-t_1}\right],\quad \Delta_0=\left[\beta_1^{t_2-t_1},+\infty\right),$$

$$\widehat{\lambda}_1=\begin{cases}\dfrac{\beta_{m+1}^{\tau-t_2}-{\beta_m}^{\tau-t_2}}{\beta_{m+1}^{t_1-t_2}-\beta_m^{t_1-t_2}}, & \dfrac{\delta_2^2}{\delta_1^2}\in\Delta_m,\ m\ge 1,\\[2ex] \beta_1^{\tau-t_1}, & \dfrac{\delta_2^2}{\delta_1^2}\in\Delta_0,\end{cases}$$

$$\widehat{\lambda}_2=\begin{cases}\dfrac{\beta_m^{\tau-t_1}-\beta_{m+1}^{\tau-t_1}}{\beta_m^{t_2-t_1}-\beta_{m+1}^{t_2-t_1}} & \dfrac{\delta_2^2}{\delta_1^2}\in\Delta_m,\ m\ge 1,\\[2ex] 0, & \dfrac{\delta_2^2}{\delta_1^2}\in\Delta_0.\end{cases}$$

Theorem 4.1 [5]: *For all $\delta_1,\delta_2>0$ the following equality*

$$E_\tau(L_2(\mathbb{B}^d),\delta_1,\delta_2)=\sqrt{\widehat{\lambda}_1\delta_1^2+\widehat{\lambda}_2\delta_2^2}$$

holds. Moreover, the method

$$\widehat{m}(y_1,y_2)=\sum_{s=1}^{\infty}\sum_{k=0}^{\infty}a_{sk}^{\tau/2}\sum_{j=1}^{a_k}\frac{\widehat{\lambda}_1a_{sk}^{t_1/2}y_{1skj}+\widehat{\lambda}_2a_{sk}^{t_2/2}y_{2skj}}{\widehat{\lambda}_1a_{sk}^{t_1}+\widehat{\lambda}_2a_{sk}^{t_2}}Y_{skj},$$

where y_{1skj},y_{2skj} are the Fourier coefficients of y_1 and y_2, is optimal.

It appears that in this case we also have a family of optimal recovery methods depending on two parameters.

Put

$$\widehat{N}_1 = \widehat{N}_1(\delta_1, \delta_2) = \begin{cases} \max\{k : \beta_k^{t_2-\tau} \le \widehat{\lambda}_2,\ k \in \mathbb{N}\}, & \beta_1^{t_2-\tau} \le \widehat{\lambda}_2, \\ 0, & \beta_1^{t_2-\tau} > \widehat{\lambda}_2, \end{cases}$$

$$\widehat{N}_2 = \widehat{N}_2(\delta_1, \delta_2) = \min\{k : \beta_k^{\tau-t_1} \le \widehat{\lambda}_1,\ k \in \mathbb{N}\}.$$

Theorem 4.2: *For all $1 \le N_1 \le \widehat{N}_1$ and $\widehat{N}_2 \le N_2$ the methods*

$$\begin{aligned}\widehat{m}_{N_1,N_2}(y_1, y_2) = \sum_{m=1}^{N_1} \beta_m^{(\tau-t_2)/2} \sum_{j=1}^{a_{k_m}} y_{2mj} X_{mj} \\ + \sum_{m=N_1+1}^{N_2-1} \beta_m^{\tau/2} \sum_{j=1}^{a_{k_m}} \frac{\widehat{\lambda}_1 \beta_m^{t_1/2} y_{1mj} + \widehat{\lambda}_2 \beta_m^{t_2/2} y_{2mj}}{\widehat{\lambda}_1 \beta_m^{t_1} + \widehat{\lambda}_2 \beta_m^{t_2}} X_{mj} \\ + \sum_{m=N_2}^{\infty} \beta_m^{(\tau-t_1)/2} \sum_{j=1}^{a_{k_m}} y_{1mj} X_{mj},\end{aligned}$$

where $X_{mj} = Y_{s_m k_m j}$ and y_{1mj}, y_{2mj} are the Fourier coefficients of y_1, y_2 for the orthonormal basis $\{X_{mj}\}$, are optimal.

We give the scheme of proof of this theorem. First, we consider the extremal problem

$$\|u(\tau, x)\|^2_{L_2(\mathbb{B}^d)} \to \max, \quad \|u(t_j, x)\|^2_{L_2(\mathbb{B}^d)} \le \delta_j^2,\ j = 1, 2, \quad f \in L_2(\mathbb{B}^d). \tag{8}$$

We have

$$\|u(\tau, x)\|^2_{L_2(\mathbb{B}^d)} = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} e^{-2(\mu_s^{(p)})^\alpha t} \sum_{j=1}^{a_k} c_{skj}^2,$$

where c_{skj} are Fourier coefficients of f. Put

$$\alpha_m = \sum_{j=1}^{a_{k_m}} c^2_{s_m k_m j}.$$

Then problem (8) we can be rewritten in the form

$$\sum_{m=1}^{\infty} \beta_m^{\tau} \alpha_m \to \max, \quad \sum_{m=1}^{\infty} \beta_m^{t_j} \alpha_m \le \delta_j^2,\ j = 1, 2, \quad \alpha_m \ge 0.$$

Next, we show that for all $1 \le N_1 \le \widehat{N}_1$ and $\widehat{N}_2 \le N_2$ the value of this problem coincides with the value of the extremal problem

$$\sum_{m=1}^{\infty} \beta_m^{\tau} \alpha_m \to \max, \quad \sum_{m=N_1+1}^{\infty} \beta_m^{t_1} \alpha_m \le \delta_1^2, \quad \sum_{m=1}^{N_2-1} \beta_m^{t_2} \alpha_m \le \delta_2^2 \quad \alpha_m \ge 0.$$

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