OPTIMAL RECOVERY OF PERIODIC FUNCTIONS FROM FOURIER COEFFICIENTS GIVEN WITH AN ERROR

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Abstract. We construct optimal methods of recovery of 2π-periodic functions analytic in a strip and its derivatives at a point $t \in [0, 2\pi)$, using information about the Fourier coefficients given with an error in the uniform norm. The same problem is solved for the Sobolev space $\widetilde{W}_2^r$.

1. Introduction

Let X and Y be linear spaces over the field $K = \mathbb{R}$ or $\mathbb{C}$, $W \subset X$ and $U \subset Y$ balanced convex sets and $I\colon W \to Y$ a linear operator. Denote by X' the set of all linear functionals on X. We consider the problem of optimal recovery of $< x', x >$ where $x' \in X'$ and $x \in W$, using information about approximate values of the operator I. A method of recovery is any function $\varphi\colon Y \to K$. The value

$$e(x', I, W, U) := \inf_{\varphi} \sup_{x \in W} \sup_{\substack{y \in Y \\ Ix - y \in U}} |< x', x > - \varphi(y)| \tag{1}$$

is called the intrinsic error in the recovery problem. Any φ_0 for which

$$e(x', I, W, U) = \sup_{x \in W} \sup_{\substack{y \in Y \\ Ix - y \in U}} |< x', x > - \varphi_0(y)|$$

is said to be an optimal method.

Many examples and other settings of optimal recovery problems can be found in [1]–[6]. It follows from Magaril-Il'yaev and Osipenko [6] that there is a linear optimal method $\varphi_0(y) = < y', y >$, $y' \in Y'$, and the following equality

$$e(x', I, W, U) = \sup_{\substack{x \in W \\ Ix \in U}} |< x', x >| \tag{2}$$

holds. On the other hand, since U and W are balanced, we have

$$\begin{aligned} e(x', I, W, U) &= \inf_{y' \in Y'} \sup_{\substack{x \in W \\ z \in U}} |< x', x > - < y', Ix - z >| \\ &= \inf_{y' \in Y'} \left(\sup_{x \in W} |< x', x > - < y', Ix >| + \sup_{z \in U} |< y', z >| \right). \end{aligned} \tag{3}$$

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In this paper, we consider the problem of optimal recovery of 2π-periodic functions analytic in a strip and its derivatives from the Hardy–Sobolev and Bergman–Sobolev spaces based on the information about Fourier coefficients given with an error in the uniform norm. We also obtain an optimal method of recovery in the analogous problem for the Sobolev space $\widetilde{W}_2^r$.

A similar problem for the estimation of functions in the L_2-norm was considered in Melkman and Micchelli [4]. The case when the l_2-norm is used to measure the error in the Fourier coefficients was analyzed by Micchelli and Rivlin [1]. In Boyanov [7] the problem of optimal recovery of periodic functions from the Sobolev space $\widetilde{W}_q^r$, $1 \le q \le \infty$, was solved for the case when the Fourier coefficients are known exactly.

2. Optimal recovery in Hilbert spaces from inaccurate Fourier coefficients

Let X be a Hilbert space and $e_1, e_2, \dots$ a complete orthonormal system in X. For $x \in X$ denote by $x_j := (x, e_j)$ the Fourier coefficients of x. Consider the problem (1) for $W = BX := \{\, x \in X : \|x\| \le 1 \,\}$, $< x', x >= (x, f)$, $f \in X$, $|f_j| > 0$, $j = 1, 2, \dots$, $Ix = (x_1, \dots, x_n)$ and

$$U = \{\, y = (y_1, \dots, y_n) : |y_j| \le \delta_j,\ j = 1, \dots, n \,\}.$$

Thus we consider the problem of optimal recovery of the linear functional (x, f) from approximate Fourier coefficients $(\tilde{x}_1, \dots, \tilde{x}_n)$ such that

$$|x_j - \tilde{x}_j| \le \delta_j, \quad j = 1, \dots, n.$$

In this case the intrinsic error will be denoted by $e(f, I, BX, \delta)$.

For $a \in \mathbb{R}$ put

$$a_+ := \begin{cases} a, & a > 0 \\ 0, & a \le 0. \end{cases}$$

Theorem 1. *Let $\lambda \in (0, \|f\|]$ be a solution of the equation*

$$\|f\|^2 - \sum_{j=1}^{n} \left(|f_j|^2 - \lambda^2 \delta_j^2\right)_+ - \lambda^2 = 0. \tag{4}$$

Then

$$(x, f) \approx \sum_{j=1}^{n} \left(1 - \lambda \delta_j |f_j|^{-1}\right)_+ \overline{f}_j \tilde{x}_j \tag{5}$$

is an optimal method of recovery and

$$e(f, I, BX, \delta) = \lambda + \sum_{j=1}^{n} \delta_j \left(|f_j| - \lambda \delta_j\right)_+ .$$

Proof. First we show that the equation (4) has a solution $\lambda \in (0, \|f\|]$. Denote by $\varphi(\lambda)$ the function on the left hand side of (4). This function is continuous for all $\lambda \geq 0$. Moreover,

$$\varphi(0) = \|f\|^2 - \sum_{j=1}^{n} |f_j|^2 > 0.$$

Since $\varphi(\|f\|) < 0$, there exists a $\lambda \in (0, \|f\|]$ which is a solution of (4).

For such λ consider the method (5). In view of (3) we have

$$e(f, I, BX, \delta) \leq \sup_{x \in BX} \left| (x, f) - \sum_{j=1}^{n} \left(1 - \lambda\delta_j |f_j|^{-1}\right)_+ \overline{f}_j x_j \right|$$
$$+ \sum_{j=1}^{n} \delta_j |f_j| \left(1 - \lambda\delta_j |f_j|^{-1}\right)_+ = \sup_{x \in BX} (x, f_\lambda) + \sum_{j=1}^{n} \delta_j \left(|f_j| - \lambda\delta_j\right)_+$$

where

$$(f_\lambda)_j = \begin{cases} f_j, & j \geq n+1 \\ f_j - f_j \left(1 - \lambda\delta_j |f_j|^{-1}\right)_+, & 1 \leq j \leq n. \end{cases}$$

It can be easily shown that

$$\|f_\lambda\|^2 = \|f\|^2 - \sum_{j=1}^{n} \left(|f_j|^2 - \lambda^2\delta_j^2\right)_+ = \lambda^2.$$

Consequently

$$e(f, I, BX, \delta) \leq \lambda + \sum_{j=1}^{n} \delta_j \left(|f_j| - \lambda\delta_j\right)_+ .$$

Put

$$x_0 := \frac{f_\lambda}{\|f_\lambda\|} = \lambda^{-1} f_\lambda.$$

Let $1 \leq j \leq n$. If $1 - \lambda\delta_j |f_j|^{-1} > 0$ then

$$|(x_0)_j| = \lambda^{-1} |(f_\lambda)_j| = \delta_j.$$

If $1 - \lambda\delta_j |f_j|^{-1} \leq 0$ then

$$|(x_0)_j| = \lambda^{-1} |f_j| \leq \delta_j.$$

Thus $Ix_0 \in U$. Using (2) we obtain

$$e(f, I, BX, \delta) \geq |(x_0, f)| = \lambda^{-1} \left(\|f\|^2 - \sum_{j=1}^{n} |f_j|^2 \left(1 - \lambda\delta_j |f_j|^{-1}\right)_+ \right)$$
$$= \lambda^{-1} \left(\|f\|^2 - \sum_{j=1}^{n} \left(|f_j| + \lambda\delta_j\right)\left(|f_j| - \lambda\delta_j\right)_+ + \lambda \sum_{j=1}^{n} \delta_j \left(|f_j| - \lambda\delta_j\right)_+ \right)$$
$$= \lambda + \sum_{j=1}^{n} \delta_j \left(|f_j| - \lambda\delta_j\right)_+ .$$

This completes the proof of the theorem.

Now let $\delta_j = \delta\lambda_j$, $\lambda_j > 0$, $j = 1, \dots, n$, and $\delta \geq 0$.

Theorem 2. *Suppose that*

$$|f_1|\lambda_1^{-1} \ge \cdots \ge |f_n|\lambda_n^{-1}.$$

Set

$$\mu_k := \left(\sum_{j=1}^{k} \lambda_j^2 + |f_k|^{-2}\lambda_k^2 \sum_{j=k+1}^{\infty} |f_j|^2 \right)^{-1/2}, \quad k = 1, \dots, n,$$

$\mu_0 := +\infty$, $\mu_{n+1} := 0$, *and* $\Delta_k := [\mu_{k+1}, \mu_k)$, $k = 0, \dots, n$. *Then for* $\delta \in \Delta_k$, $0 \le k \le n$, *the method*

$$(x, f) \approx \sum_{j=1}^{k} \left(1 - \delta \frac{\lambda_j}{|f_j|} \sqrt{\frac{\sum_{j=k+1}^{\infty} |f_j|^2}{1 - \delta^2 \sum_{j=1}^{k} \lambda_j^2}} \right) \overline{f}_j \tilde{x}_j$$

is optimal and

$$e(f, I, BX, \delta) = \sqrt{\sum_{j=k+1}^{\infty} |f_j|^2} \sqrt{1 - \delta^2 \sum_{j=1}^{k} \lambda_j^2} + \delta \sum_{j=1}^{k} \lambda_j |f_j|.$$

Proof. The equation (4) now takes the following form

$$\|f\|^2 - \sum_{j=1}^{n} \left(|f_j|^2 - \lambda^2 \delta^2 \lambda_j^2 \right)_+ - \lambda^2 = 0. \tag{6}$$

If $\delta = 0$, then the solution of (6) is evident and the theorem follows from Theorem 1 immediately. If $\delta > 0$, then (6) is equivalent to the equation

$$\frac{c^2}{\|f\|^2 - \sum_{j=1}^{n} \left(|f_j|^2 - c^2 \lambda_j^2 \right)_+} = \delta^2 \tag{7}$$

where $c = \lambda\delta$. Denote by $\varphi(c)$ the function on the left hand side of (7). It is easy to show that $\varphi(c)$ is monotonically increasing for $c \ge 0$. Furthemore,

$$\varphi\left(|f_k|\lambda_k^{-1} \right) = \mu_k^2, \quad k = 1, \dots, n.$$

Hence for $\delta \in \Delta_k$, $0 \le k \le n$,

$$c = \delta \sqrt{\frac{\sum_{j=k+1}^{\infty} |f_j|^2}{1 - \delta^2 \sum_{j=1}^{k} \lambda_j^2}}$$

is the solution of (7). Now the theorem follows from Theorem 1.

For $\lambda_1 = \cdots = \lambda_n = 1$, Theorem 2 was proved in [8] using more complicated arguments.

Denote by L the linear space of vectors $x=(x_1,x_2,\dots)$, $x_j\in\mathbb{C}$, which satisfy the condition

$$\sum_{j=1}^{\infty}\gamma_j|x_j|^2<\infty$$

where $\gamma_1\geq 0$ and $\gamma_j>0$, $j>1$. Let x' be the linear functional on L defined by the following equality

$$\langle x',x\rangle:=\sum_{j=1}^{\infty}x_j\overline{f}_j$$

where $|f_j|>0$, $j>1$, and

$$\sum_{j=2}^{\infty}\gamma_j^{-1}|f_j|^2<\infty.$$

Consider the problem of optimal recovery of the functional x' on the set $BL:=\{\,x\in L:\sum_{j=1}^{\infty}\gamma_j|x_j|^2\leq 1\,\}$ from information $(\tilde{x}_1,\dots,\tilde{x}_n)$ such that

$$|x_j-\tilde{x}_j|\leq\delta\lambda_j,\quad \lambda_j>0,\quad j=1,\dots,n.$$

Put

$$m:=\begin{cases}1, & \gamma_1f_1\neq 0\\ 2, & \gamma_1f_1=0.\end{cases}$$

Theorem 3. *Suppose that*

$$\frac{|f_m|}{\lambda_m\gamma_m}\geq\dots\geq\frac{|f_n|}{\lambda_n\gamma_n}. \tag{8}$$

Set

$$\mu_{km}:=\left(\sum_{j=m}^{k}\gamma_j\lambda_j^2+\gamma_k^2|f_k|^{-2}\lambda_k^2\sum_{j=k+1}^{\infty}\gamma_j^{-1}|f_j|^2\right)^{-1/2},\quad k=m,\dots,n,$$

$\mu_{m-1,m}:=+\infty$, $\mu_{n+1,m}:=0$, *and* $\Delta_{km}:=[\mu_{k+1,m},\mu_{km})$, $k=m-1,\dots,n$. *Then for* $\delta\in\Delta_{km}$, $m-1\leq k\leq n$, *the method*

$$\langle x',x\rangle\approx(m-1)\overline{f}_1\tilde{x}_1+\sum_{j=m}^{k}\nu_{jm}\overline{f}_j\tilde{x}_j, \tag{9}$$

where

$$\nu_{jm}=1-\delta\frac{\gamma_j\lambda_j}{|f_j|}\sqrt{\frac{\sum_{j=k+1}^{\infty}\gamma_j^{-1}|f_j|^2}{1-\delta^2\sum_{j=m}^{k}\gamma_j\lambda_j^2}},$$

is optimal and

$$e(x',I,BL,\delta)=\sqrt{\sum_{j=k+1}^{\infty}\gamma_j^{-1}|f_j|^2}\sqrt{1-\delta^2\sum_{j=m}^{k}\gamma_j\lambda_j^2}+\delta\sum_{j=m}^{k}\lambda_j|f_j|.$$

Proof. Consider the case $m = 1$. Then L is a Hilbert space with the inner product

$$(x, y)_L := \sum_{j=1}^{\infty} \gamma_j x_j \overline{y}_j.$$

The vectors $e_1, e_2, \dots,$

$$(e_j)_s := \begin{cases} 0, & s \neq j, \\ \gamma_j^{-1/2}, & s = j, \end{cases}$$

form a complete orthonormal basis in L. The Fourier coefficients of x are equal to $(x, e_j) = \sqrt{\gamma_j} x_j$. Now we can use Theorem 2 in which we have to replace λ_j and f_j by $\gamma_j^{1/2}\lambda_j$ and $\gamma_j^{-1/2} f_j$, respectively.

Suppose that $\gamma_1 = 0$. Denote by L_0 the space of all vectors $x \in L$ for which $x_1 = 0$. The space L_0 is a Hilbert space with the inner product

$$(x, y)_{L_0} := \sum_{j=2}^{\infty} \gamma_j x_j \overline{y}_j.$$

From Theorem 2 it follows that the method

$$\langle x', x\rangle \approx \sum_{j=2}^{k} \nu_{jm} \overline{f}_j \tilde{x}_j$$

is optimal for the set BL_0, and

$$e(x', I, BL_0, \delta) = \sqrt{\sum_{j=k+1}^{\infty} \gamma_j^{-1} |f_j|^2} \sqrt{1 - \delta^2 \sum_{j=2}^{k} \gamma_j \lambda_j^2} + \delta \sum_{j=2}^{k} \lambda_j |f_j|.$$

From (2) we have

$$e(x', I, BL, \delta) = \delta\lambda_1 |f_1| + e(x', I, BL_0, \delta). \tag{10}$$

On the other hand, from (3) it follows that for the method (9)

$$\begin{aligned} e(x', I, BL, \delta) \leq \sup_{x \in BL_0} \left| \langle x', x\rangle - \sum_{j=2}^{k} \nu_{jm} \overline{f}_j x_j \right| + \delta\lambda_1 |f_1| + \sup_{|z_j| \leq \delta\lambda_j} \left| \sum_{j=2}^{k} \nu_{jm} \overline{f}_j z_j \right| \\ = \delta\lambda_1 |f_1| + e(x', I, BL_0, \delta). \end{aligned}$$

In view of (10) the method (9) is optimal for the set BL.

Now assume that $f_1 = 0$. Since from (2)

$$e(x', I, BL, \delta) = e(x', I, BL_0, \delta),$$

it sufficies to construct an optimal method for the set BL_0. It can be immediately obtained from Theorem 2. The theorem is proved.

3. Optimal recovery in Hardy–Sobolev and Bergman–Sobolev spaces

Let W be a shift invariant class of sufficiently smooth and 2π-periodic functions. Consider the problem of optimal recovery of $f^{(s)}(t)$, $t \in [0, 2\pi)$, $f \in W$, using information about the Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\, dt, \quad |k| \le n,$$

given with error at most δ in the uniform norm, i.e., by $\tilde{c}_k$ such that

$$|c_k - \tilde{c}_k| \le \delta, \quad |k| \le n.$$

Denote by $e_{ns}(W, \delta)$ the intrinsic error for this problem (from (2) it follows that it does not depend on t).

Let $\widetilde{H}_{2,\beta}$ be the space of all 2π-periodic functions analytic in the strip $S_\beta := \{\, z \in \mathbb{C} : |\operatorname{Im} z| < \beta \,\}$ which satisfy the condition

$$\|f\|_{\widetilde{H}_{2,\beta}} := \sup_{0 \le \eta < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} \left(|f(t + i\eta)|^2 + |f(t - i\eta)|^2 \right) dt \right)^{1/2} < \infty.$$

The Hardy–Sobolev space $\widetilde{H}^r_{2,\beta}$ is the set of all 2π-periodic functions analytic in the strip S_β for which $f^{(r)} \in \widetilde{H}_{2,\beta}$. Set

$$B\widetilde{H}^r_{2,\beta} := \{\, f \in \widetilde{H}^r_{2,\beta} : \|f^{(r)}\|_{\widetilde{H}_{2,\beta}} \le 1 \,\}, \quad r = 0, 1, \dots .$$

Functions from $\widetilde{H}_{2,\beta}$ have finite boundary values almost everywhere and the space $\widetilde{H}_{2,\beta}$ can be considered as a Hilbert space with the inner product

$$(f, g)_{\widetilde{H}_{2,\beta}} := \frac{1}{4\pi} \int_0^{2\pi} \left(f(t + i\beta)\overline{g(t + i\beta)} + f(t - i\beta)\overline{g(t - i\beta)} \right) dt.$$

It is easy to verify that the functions $e_j(z) := e^{ijz}$, $j = 0, \pm 1, \dots$ form a complete orthogonal basis in $\widetilde{H}_{2,\beta}$ and $\|e_j\|^2_{\widetilde{H}_{2,\beta}} = \cosh 2j\beta$. Thus $f \in B\widetilde{H}^r_{2,\beta}$ iff

$$f(z) = \sum_{j=-\infty}^{+\infty} c_j e^{ijz}$$

and

$$\sum_{j=-\infty}^{+\infty} |c_j|^2 j^{2r} \cosh 2j\beta \le 1.$$

For $p = \{p_j\}_{-\infty}^{+\infty}$, $p_j > 0$, we introduce the following notation

$$\mu_{kr}(p, s) := \left(\sum_{|j| \le k} j^{2r} p_j + k^{4r-2s} p_k^2 \sum_{|j| > k} j^{2(s-r)} p_j^{-1} \right)^{-1/2}, \quad 1 \le k \le n,$$

$$\mu_{n+1,r}(p, s) := 0, \quad \mu_{00}(p, s) := \left(\sum_{|j| \ge 0} p_j^{-1} \right)^{-1/2}, \quad \mu_{0r}(p, s) := +\infty, \ r \ge 1,$$

$$\Delta_{kr}(p, s) := \left[\mu_{k+1,r}(p, s), \mu_{kr}(p, s) \right), \quad 0 \le k \le n, \ r \ge 0,$$

$$\Delta_{-1,0}(p, s) := \left[\mu_{00}(p, s), +\infty \right).$$

Using Theorem 3 with $\overline{f}_j=(ij)^s e^{ijt}$, $\lambda_j=1$ and $\gamma_j=j^{2r}\cosh 2j\beta$, we obtain the following result.

Theorem 4. *Let r and s be nonnegative integers such that $0\le s\le 2r$. Put $p_j=\cosh 2j\beta$, $j=0,\pm1,\dots$. For $\delta\in\Delta_{kr}(p,s)$ the method*

$$f^{(s)}(t)\approx\sum_{|j|\le k}\nu_{jk}(p,s,\delta)\tilde c_j(ij)^s e^{ijt}, \tag{11}$$

where

$$\nu_{jk}(p,s,\delta)=1-\delta|j|^{2r-s}p_j\sqrt{\frac{\sum_{|j|>k}j^{2(s-r)}p_j^{-1}}{1-\delta^2\sum_{|j|\le k}j^{2r}p_j}},$$

is optimal for the class $B\widetilde H^r_{2,\beta}$, and

$$e_{ns}(B\widetilde H^r_{2,\beta},\delta)=E_{kr}(p,s,\delta):=\sqrt{\sum_{|j|>k}j^{2(s-r)}p_j^{-1}}\sqrt{1-\delta^2\sum_{|j|\le k}j^{2r}p_j}+\delta\sum_{|j|\le k}|j|^s.$$

Denote by $\widetilde A_{2,\beta}$ the space of all 2π-periodic functions analytic in the strip S_β which satisfy the condition

$$\|f\|_{\widetilde A_{2,\beta}}:=\left(\frac{1}{4\pi\beta}\int_0^{2\pi}\int_{-\beta}^{\beta}|f(t+i\eta)|^2\,dt\,d\eta\right)^{1/2}<\infty.$$

The Bergman–Sobolev space $\widetilde A^r_{2,\beta}$ is the set of all 2π-periodic functions analytic in the strip S_β for which $f^{(r)}\in\widetilde A_{2,\beta}$. Set

$$B\widetilde A^r_{2,\beta}:=\{\,f\in\widetilde A^r_{2,\beta}:\|f^{(r)}\|_{\widetilde A_{2,\beta}}\le 1\,\},\quad r=0,1,\dots\ .$$

Consider the problem of optimal recovery of $f^{(s)}(t)$ for the class $B\widetilde A^r_{2,\beta}$.

$\widetilde A_{2,\beta}$ is a Hilbert space with the inner product

$$(f,g)_{\widetilde A_{2,\beta}}:=\frac{1}{4\pi\beta}\int_0^{2\pi}\int_{-\beta}^{\beta}f(t+i\eta)\overline{g(t+i\eta)}\,dt\,d\eta.$$

It can be easily shown that the functions $e_j(z)$, $j=0,\pm1,\dots$ form a complete orthogonal basis in $\widetilde A_{2,\beta}$ and

$$\|e_0\|_{\widetilde A_{2,\beta}}=1,\qquad \|e_j\|^2_{\widetilde A_{2,\beta}}=\frac{\sinh 2j\beta}{2j\beta},\quad j=\pm1,\pm2,\dots\ .$$

Therefore $f\in B\widetilde A^r_{2,\beta}$ iff

$$f(z)=\sum_{j=-\infty}^{+\infty}c_je^{ijz}$$

and

$$\sum_{j=-\infty}^{+\infty}|c_j|^2j^{2r}\|e_j\|^2_{\widetilde A_{2,\beta}}\le 1.$$

Analogously to Theorem 4 we have

Theorem 5. *Let $0 \le s \le 2r$. Put*

$$p_0 = 1, \qquad p_j = \frac{\sinh 2j\beta}{2j\beta}, \quad j = \pm 1, \pm 2, \dots .$$

For $\delta \in \Delta_{kr}(p, s)$ the method (11) *is an optimal method for the class $B\widetilde{A}^r_{2,\beta}$ and*

$$e_{ns}(B\widetilde{A}^r_{2,\beta}, \delta) = E_{kr}(p, s, \delta).$$

Remark. We need the condition $0 \le s \le 2r$ to satisfy (8). For $s > 2r$ optimal methods of recovery for the classes $B\widetilde{H}^r_{2,\beta}$ and $B\widetilde{A}^r_{2,\beta}$ can be constructed by Theorem 1.

Almost the same arguments as in Theorem 4 and Theorem 5 enable us to obtain an optimal method of recovery of $f^{(s)}(t)$, $0 \le s \le r-1$, for the Sobolev class $B\widetilde{W}^r_2$ which is the set of all real-valued 2π-periodic functions such that $f^{(r-1)}$ is absolutely continuous and

$$\frac{1}{2\pi} \int_0^{2\pi} |f^{(r)}(t)|^2 \, dt \le 1.$$

Theorem 6. *Let $0 \le s \le r-1$. Put $p_j = 1$, $j = 0, \pm 1, \dots$. For $\delta \in \Delta_{kr}(p, s)$ the method* (11) *is an optimal method for the class $B\widetilde{W}^r_2$ and*

$$e_{ns}(B\widetilde{W}^r_2, \delta) = E_{kr}(p, s, \delta).$$

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