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On some Carlson-type inequalities

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Abstract. We find the sharp constant in the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T)} \leqslant K\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T)}\biggl(\sum_{j=1}^{d}\|\varphi_j(\cdot)x(\cdot)\|^r_{L_r(T)}\biggr)^{(1-\gamma)/r},$$

where T is a cone in $\mathbb{R}^d$ and the weights $w(\cdot)$, $w_0(\cdot)$ and $\varphi_j(\cdot)$, $j=1,\dots,d$, are homogeneous measurable functions. We also obtain similar inequalities for some differential operators.

Bibliography: 7 titles.

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Introduction

The well-known Carlson inequality [1]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \leqslant \sqrt{\pi}\|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)}\|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)}, \qquad \mathbb{R}_+=[0,+\infty),$$

has been generalized by many authors (for instance, see [2]–[7]). We state one of its generalizations which we use below.

Consider the spherical system of coordinates in $\mathbb{R}^d$:

$$\begin{aligned} t_1&=\rho\cos\omega_1,\\ t_2&=\rho\sin\omega_1\cos\omega_2,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ t_{d-1}&=\rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\cos\omega_{d-1},\\ t_d&=\rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}. \end{aligned}$$

For a function $f(t)$, $t\in\mathbb{R}^d$, set

$$\widetilde{f}(\omega)=|f(\cos\omega_1,\dots,\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1})|, \qquad \omega=(\omega_1,\dots,\omega_{d-1}).$$

We introduce the following notation:

$$P=\{(p,q,r)\colon 1\leqslant q<p,r\}, \qquad P_1=\{(p,q,r)\colon 1\leqslant q=r<p\},$$
$$P_2=\{(p,q,r)\colon 1\leqslant q=p<r\}.$$

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Let $|w(\cdot)|$, $|w_0(\cdot)|$ and $|w_1(\cdot)|$ be homogeneous measurable functions on $\mathbb{R}^d$ of degrees θ, θ_0 and θ_1, respectively, let T be a cone in $\mathbb{R}^d$ and Ω be the range of ω for $t \in T$. Since T is a cone, Ω is independent of ρ. Set

$$J(\omega) = \sin^{d-2}\omega_1 \sin^{d-3}\omega_2 \cdots \sin\omega_{d-2}.$$

The following result is a consequence of [7] (Corollary 4 for $n = 1$).

Theorem 1. *Let $w(\cdot), w_0(\cdot), w_1(\cdot) \neq 0$ for almost all $t \in T$, let $(p, q, r) \in P \cup P_1 \cup P_2$, $\gamma = (\theta_1 - \theta - d(1/q - 1/r))/(\theta_1 - \theta_0 + d(1/r - 1/p))$ and $\gamma \in (0, 1)$. Assume that*

$$I = \int_\Omega \left(\frac{\widetilde{w}(\omega)}{\widetilde{w}_0^\gamma(\omega)\widetilde{w}_1(\omega)^{(1-\gamma)}} \right)^{\widetilde{q}} J(\omega)\, d\omega < \infty,$$

where

$$\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(T)$ and $w_1(\cdot)x(\cdot) \in L_r(T)$ the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T)} \leqslant K \|w_0(\cdot)x(\cdot)\|_{L_p(T)}^\gamma \|w_1(\cdot)x(\cdot)\|_{L_r(T)}^{1-\gamma}$$

holds, where

$$K = \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/r} \left(\frac{B(\widetilde{q}\gamma/p, \widetilde{q}(1-\gamma)/r) I}{|\theta_1 - \theta_0 + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)} \right)^{1/\widetilde{q}}$$

and $B(\,\cdot\,, \cdot\,)$ is the Euler beta function.

For $(p, q, r) \in P$ Theorem 1 was proved in [4] (also see Corollary 4 in [5]).

By the properties of the beta function

$$B\left(\frac{\widetilde{q}\gamma}{p}, \frac{\widetilde{q}(1-\gamma)}{r} \right) = \frac{\gamma r + (1-\gamma)p}{\gamma r} B\left(\frac{\widetilde{q}\gamma}{p} + 1, \frac{\widetilde{q}(1-\gamma)}{r} \right).$$

Hence K has the following expression:

$$K = \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/r} \left(\frac{B(\widetilde{q}\gamma/p + 1, \widetilde{q}(1-\gamma)/r) I}{r|\theta_1 - \theta - d(1/q - 1/r)|} \right)^{1/\widetilde{q}}.$$

§ 1. Sharp inequalities on cones in $\mathbb{R}^d$

Let

$$w_1(t) = \left(\sum_{j=1}^n |\varphi_j(t)|^r \right)^{1/r}.$$

Then

$$\|w_1(\cdot)x(\cdot)\|_{L_r(T)}^r = \sum_{j=1}^n \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T)}^r.$$

Thus we can state Theorem 1 as follows.

Theorem 2. *Let $|w(\cdot)|$ and $|w_0(\cdot)|$ be homogenous measurable functions of degrees θ and θ_0, respectively, and $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogeneous measurable functions of degree θ_1. In addition, assume that $w(t), w_0(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$, and let $(p,q,r) \in P \cup P_1 \cup P_2$ and $\gamma \in (0,1)$ be as in Theorem 1. Let*

$$I = \int_\Omega \left(\frac{\widetilde{w}(\omega)}{\widetilde{w}_0^\gamma(\omega)(\sum_{j=1}^n \widetilde{\varphi}_j^r(\omega))^{(1-\gamma)/r}} \right)^{\widetilde{q}} J(\omega)\, d\omega < \infty.$$

Then the following sharp inequality holds for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(T)$ and $\varphi_j(\cdot)x(\cdot) \in L_r(T)$, $j=1,\dots,n$:

$$\|w(\cdot)x(\cdot)\|_{L_q(T)} \leqslant K \|w_0(\cdot)x(\cdot)\|^\gamma_{L_p(T)} \left(\sum_{j=1}^n \|\varphi_j(\cdot)x(\cdot)\|^r_{L_r(T)} \right)^{(1-\gamma)/r}.$$

Corollary 1. *Let $|w(\cdot)|$ and $|w_0(\cdot)|$ be homogeneous measurable functions of degrees $d(1-1/q)$ and $d-(\lambda+d)/p$, respectively, and let $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogeneous measurable functions of degree $d+(\mu-d)/r$, where $\lambda,\mu>0$. Also assume that $w(t), w_1(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$, and let $(p,q,r) \in P \cup P_1 \cup P_2$. Let*

$$I = \int_\Omega \left(\frac{\widetilde{w}(\omega)}{\widetilde{w}_0^{p\alpha}(\omega)(\sum_{j=1}^n \widetilde{\varphi}_j^r(\omega))^{\beta}} \right)^{1/(1/q-\alpha-\beta)} J(\omega)\, d\omega < \infty,$$

where

$$\alpha = \frac{\mu}{p\mu + r\lambda} \quad \text{and} \quad \beta = \frac{\lambda}{p\mu + r\lambda}.$$

Then the following sharp inequality holds for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(T)$ and $\varphi_j(\cdot)x(\cdot) \in L_r(T)$, $j=1,\dots,n$:

$$\|w(\cdot)x(\cdot)\|_{L_q(T)} \leqslant \widehat{K} \|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(T)} \left(\sum_{j=1}^n \|\varphi_j(\cdot)x(\cdot)\|^r_{L_r(T)} \right)^{\beta},$$

where

$$\widehat{K} = \frac{1}{(p\alpha)^\alpha (r\beta)^\beta} \left(\frac{I}{\lambda+\mu} B\left(\frac{\alpha}{1/q-\alpha-\beta}, \frac{\beta}{1/q-\alpha-\beta} \right) \right)^{1/q-\alpha-\beta}.$$

In the case when $T = \mathbb{R}^d_+$, $q=1$, $p,r>1$, $w(t) \equiv 1$, $n=1$,

$$w_0(t) = W^{1-(\lambda+1)/p}(t) \quad \text{and} \quad w_1(t) = W^{1+(\mu-1)/r}(t),$$

where $W(\cdot)$ is homogeneous of degree d, the result of Corollary 1 was obtained in [3]. In the one-dimensional case ($d=1$), under the same assumptions as above the result in question was derived in [2].

§ 2. Sharp inequalities for differential operators

2.1. Sharp inequalities in the metric $L_2(\mathbb{R}^d)$. Let S be the Schwartz space of rapidly decaying infinitely differentiable functions on $\mathbb{R}$, S' be the corresponding space of distributions and $F\colon S'\to S'$ be the Fourier transform.

Let $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogeneous functions of degree ν and $|\psi(\cdot)|$ be homogeneous of degree η. Set

$$X_p=\bigl\{x(\cdot)\in S'\colon \varphi_j(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d),\ j=1,\dots,n,\ Fx(\cdot)\in L_p(\mathbb{R}^d)\bigr\}.$$

For functions $x(\cdot)\in X_p$ let D_j, $j=1,\dots,n$, be the operators defined by

$$D_jx(\cdot)=F^{-1}(\varphi_j(\cdot)Fx(\cdot))(\cdot),\qquad j=1,\dots,n,$$

and let

$$\Lambda x(\cdot)=F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot) \tag{2.1}$$

(we assume that the function $\psi(\cdot)$ satisfies $\psi(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d)$ for $x(\cdot)\in X_p$).

Set

$$C_p(\nu,\eta)=\widehat{\gamma}^{-\widehat{\gamma}/p}(1-\widehat{\gamma})^{-(1-\widehat{\gamma})/2}\left(\frac{B(\widehat{q}\widehat{\gamma}/p+1,\widehat{q}(1-\widehat{\gamma})/2)}{2|\nu-\eta|}\right)^{1/\widehat{q}},$$

where

$$\widehat{\gamma}=\frac{\nu-\eta}{\nu+d(1/2-1/p)}\quad\text{and}\quad \widehat{q}=\frac{1}{\widehat{\gamma}(1/2-1/p)}.$$

By Theorem 6 in [7] (similarly to Theorem 2 here) the following result holds.

Theorem 3. *Let $2<p\leqslant\infty$ and $\widehat{\gamma}\in(0,1)$. Also let*

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}}(\omega)}{(\sum_{j=1}^n\widetilde{\varphi}_j^2(\omega))^{\widehat{q}(1-\widehat{\gamma})/2}}J(\omega)\,d\omega<\infty,\qquad \Pi^{d-1}=[0,\pi]^{d-2}\times[0,2\pi].$$

Then the following sharp inequality holds:

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}\leqslant\frac{C_p(\nu,\eta)I^{1/\widehat{q}}}{(2\pi)^{d\widehat{\gamma}/2}}\|Fx(\cdot)\|^{\widehat{\gamma}}_{L_p(\mathbb{R}^d)}\left(\sum_{j=1}^n\|D_jx(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{(1-\widehat{\gamma})/2}. \tag{2.2}$$

We look at a few concrete weights. Let $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d_+$. Consider the operator D^α (derivative of order α) defined by

$$D^\alpha x(\cdot)=F^{-1}((i\xi)^\alpha Fx(\xi))(\cdot),$$

where $(i\xi)^\alpha=(i\xi_1)^{\alpha_1}\cdots(i\xi_d)^{\alpha_d}$. It is cleat that if $x(\cdot)$ is a sufficiently smooth function on $\mathbb{R}^d$, $t=(t_1,\dots,t_d)\in\mathbb{R}^d$ and $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{Z}^d_+$, then

$$D^\alpha x(t)=\frac{\partial x^{\alpha_1+\dots+\alpha_d}(t)}{\partial t_1^{\alpha_1}\cdots\partial t_d^{\alpha_d}}.$$

Set

$$\psi_\theta(\xi) = (|\xi_1|^\theta + \cdots + |\xi_d|^\theta)^{2/\theta}, \qquad \theta > 0.$$

We let $\Lambda_\theta^{\eta/2}$ denote the operator Λ defined by (2.1) for $\psi(\xi) = \psi_\theta^{\eta/2}(\xi)$. In particular, $\Lambda_2 = -\Delta$, where Δ is the Laplace operator. We obtain the sharp inequality (2.2) for $\Lambda = \Lambda_\theta^{\eta/2}$ and $D_j = D^{\nu e_j}$, $j = 1, \dots, d$, where $\{e_j\}$ is the standard basis of $\mathbb{R}^d$.

For $\varphi_j(\xi) = (i\xi_j)^\nu$ we have $\widetilde{\varphi}_j(\omega) = \widetilde{t}_j^\nu(\omega)$, where

$$\begin{aligned}
\widetilde{t}_1(\omega) &= |\cos\omega_1|,\\
\widetilde{t}_2(\omega) &= |\sin\omega_1 \cos\omega_2|,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\widetilde{t}_{d-1}(\omega) &= |\sin\omega_1 \sin\omega_2 \cdots \sin\omega_{d-2} \cos\omega_{d-1}|,\\
\widetilde{t}_d(\omega) &= |\sin\omega_1 \sin\omega_2 \cdots \sin\omega_{d-2} \sin\omega_{d-1}|.
\end{aligned}$$

Note that $\sum_{k=1}^d \widetilde{t}_k^2(\omega) = 1$.

For the quantity I in Theorem 3 we have

$$I = \int_{\Pi^{d-1}} \frac{\big(\sum_{k=1}^d \widetilde{t}_k^\theta(\omega)\big)^{\widehat{q}\eta/\theta} J(\omega)\, d\omega}{\big(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\big)^{\widehat{q}(1-\widehat{\gamma})/2}}. \tag{2.3}$$

If $\nu \leqslant 1$, then

$$\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega) \geqslant \sum_{k=1}^d \widetilde{t}_k^2(\omega) = 1. \tag{2.4}$$

On the other hand, if $\nu > 1$, then by Hölder's inequality

$$1 = \sum_{k=1}^d \widetilde{t}_k^2(\omega) \leqslant \left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{1/\nu} d^{1-1/\nu}.$$

Thus,

$$\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega) \geqslant d^{1-\nu}. \tag{2.5}$$

It follows from (2.4) and (2.5) that $I < \infty$.

Theorem 3 is proved.

From Theorem 3 we deduce the following result.

Corollary 2. *Let $2 < p \leqslant \infty$ and $\nu > \eta \geqslant 0$. Then the sharp inequality*

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{C_p(\nu,\eta) I^{1/\widehat{q}}}{(2\pi)^{d\widehat{\gamma}/2}} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}} \left(\sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2\right)^{(1-\widehat{\gamma})/2}$$

holds, where I is defined by (2.3).

In particular, for $\theta = 2$, $\nu \in \mathbb{Z}$, $2 < p \leqslant \infty$ and $\nu > \eta \geqslant 0$ we have the sharp inequality

$$\|(-\Delta)^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{C_p(\nu,\eta)I^{1/\widehat{q}}}{(2\pi)^{d\widehat{\gamma}/2}}\|Fx(\cdot)\|^{\widehat{\gamma}}_{L_p(\mathbb{R}^d)}\left(\sum_{j=1}^d\left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|^2_{L_2(\mathbb{R}^d)}\right)^{(1-\widehat{\gamma})/2}.$$

Now let $\Lambda = D^\alpha$, and let $D_j = D^{\nu e_j}$, $j = 1,\dots,d$. Then I in Theorem 3 has the form

$$I = \int_{\Pi^{d-1}} \frac{\left(\widetilde{t}_1^{\alpha_1}(\omega)\cdots\widetilde{t}_d^{\alpha_d}(\omega)\right)^{q_1} J(\omega)\,d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{q_1(1-\gamma_1)/2}}, \tag{2.6}$$

where

$$\gamma_1 = \frac{\nu-|\alpha|}{\nu+d(1/2-1/p)}, \quad q_1 = \frac{1}{\gamma_1(1/2-1/p)} \quad \text{and} \quad |\alpha| = \alpha_1+\dots+\alpha_d.$$

It follows from (2.4) and (2.5) that $I < \infty$. From Theorem 3 we deduce the following result.

Corollary 3. *Let $2 < p \leqslant \infty$ and $\nu > |\alpha| \geqslant 0$. Then the sharp inequality*

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{\widetilde{C}_p(\nu,|\alpha|)I^{1/q_1}}{(2\pi)^{d\gamma_1/2}}\|Fx(\cdot)\|^{\gamma_1}_{L_p(\mathbb{R}^d)}\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{(1-\gamma_1)/2}$$

holds, where

$$\widetilde{C}_p(\nu,|\alpha|) = \gamma_1^{-\gamma_1/p}(1-\gamma_1)^{-(1-\gamma_1)/2}\left(\frac{B(q_1\gamma_1/p+1, q_1(1-\gamma_1)/2)}{2(\nu-|\alpha|)}\right)^{1/q_1}$$

and I is defined by (2.6).

We obtain a sharp inequality for the operator $\Lambda_\theta^{\eta/2}$ in the case $p = 2$.

Theorem 4. *Let $\nu > \eta > 0$ and $0 < \theta \leqslant 2\nu$. Then the following sharp inequality holds:*

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{d^{\eta(1/\theta-1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}}\|Fx(\cdot)\|^{1-\eta/\nu}_{L_2(\mathbb{R}^d)}\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{\eta/(2\nu)}. \tag{2.7}$$

Proof. Consider the extremal problem

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \max, \qquad \|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant \delta^2, \quad \sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant 1. \tag{2.8}$$

For $0 < \varepsilon < (2\pi)^{d/(2\nu)}(d\delta^2)^{-1/(2\nu)}$ set

$$\widehat{\xi}_\varepsilon = \left(\frac{(2\pi)^d}{d\delta^2}\right)^{1/(2\nu)}(1,\dots,1) - (\varepsilon,\dots,\varepsilon) \quad \text{and} \quad B_\varepsilon = \{\xi \in \mathbb{R}^d \colon |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider the function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi)=\begin{cases}\dfrac{\delta}{\sqrt{\operatorname{mes} B_\varepsilon}}, & \xi\in B_\varepsilon,\\ 0, & \xi\notin B_\varepsilon.\end{cases}$$

Then $\|Fx_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\delta^2$ and

$$\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{\delta^2}{(2\pi)^d\operatorname{mes} B_\varepsilon}\sum_{j=1}^d\int_{B_\varepsilon}|\xi_j|^{2\nu}\,d\xi\leqslant 1.$$

Therefore, $x_\varepsilon(\cdot)$ is an admissible function in (2.8). We denote its solution by S. Then

$$S\geqslant\|\Lambda_\theta^{\eta/2}x_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{\delta^2}{(2\pi)^d\operatorname{mes} B_\varepsilon}\int_{B_\varepsilon}\psi_\theta^\eta(\xi)\,d\xi=\frac{\delta^2}{(2\pi)^d}\psi_\theta^\eta(\widetilde{\xi}_\varepsilon),\qquad \widetilde{\xi}_\varepsilon\in B_\varepsilon.$$

Letting ε tend to zero we obtain

$$S\geqslant d^{\eta(2/\theta-1/\nu)}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}. \tag{2.9}$$

Set

$$\begin{aligned}\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)&=-\|\Lambda_\theta^{\eta/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}+\lambda_1\|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)}+\lambda_2\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)\\ &=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(-\psi_\theta^\eta(\xi)+(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}\right)|Fx(\xi)|^2\,d\xi.\end{aligned}$$

Since $\theta\leqslant 2\nu$, it follows from Hölder's inequality that

$$\sum_{j=1}^d|\xi_j|^\theta\leqslant\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{\theta/(2\nu)}d^{1-\theta/(2\nu)}.$$

Setting $\rho=(|\xi_1|^\theta+\cdots+|\xi_d|^\theta)^{1/\theta}$ we obtain

$$\sum_{j=1}^d|\xi_j|^{2\nu}\geqslant\rho^{2\nu}d^{1-2\nu/\theta}.$$

Thus,

$$-\psi_\theta^\eta(\xi)+(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}\geqslant f(\rho),$$

where

$$f(\rho)=-\rho^{2\eta}+(2\pi)^d\lambda_1+\lambda_2\rho^{2\nu}d^{1-2\nu/\theta}.$$

Set

$$\lambda_1=\frac{d^{\eta(2/\theta-1/\nu)}}{(2\pi)^d}\left(1-\frac{\eta}{\nu}\right)\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu}\quad\text{and}\quad\lambda_2=d^{\eta(2/\theta-1/\nu)}\frac{\eta}{\nu}\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu-1}.$$

It is easy to see that $f(\rho)$ attains its minimum on $[0,+\infty)$ at the point

$$\rho_0=d^{(1/\theta-1/(2\nu))}\left(\frac{(2\pi)^d}{\delta^2}\right)^{1/(2\nu)},$$

and moreover, $f(\rho_0)=0$. Hence

$$-\psi_\theta^{\eta/2}(\xi)+(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}\geqslant 0.$$

Therefore, $\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)\geqslant 0$.

For each admissible function $x(\cdot)$ in (2.8) we have

$$\begin{aligned}-\|\Lambda_\theta^{\eta/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}&\geqslant-\|\Lambda_\theta^{\eta/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}+\lambda_1\big(\|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)}-\delta^2\big)\\&\quad+\lambda_2\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}-1\right)\\&=\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)-\lambda_1\delta^2-\lambda_2\geqslant-\lambda_1\delta^2-\lambda_2.\end{aligned}$$

Hence

$$S\leqslant\lambda_1\delta^2+\lambda_2=d^{\eta(2/\theta-1/\nu)}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}.$$

Taking (2.9) into account we obtain

$$S=d^{\eta(2/\theta-1/\nu)}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}.\tag{2.10}$$

Assume that $x(\cdot)\in L_2(\mathbb{R}^d)$ and $D^{\nu e_j}x(\cdot)\in L_2(\mathbb{R}^d)$, $j=1,\dots,d$. Set

$$A=\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{1/2}.$$

Then for $\varepsilon>0$ and $\widehat{x}(\cdot)=x(\cdot)/(A+\varepsilon)$ we have

$$\sum_{j=1}^d\|D^{\nu e_j}\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{A^2}{(A+\varepsilon)^2}<1\quad\text{and}\quad\|F\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}}{A+\varepsilon}.$$

It follows from (2.10) that

$$\|\Lambda_\theta^{\eta/2}\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}^d)}\leqslant\frac{d^{\eta(2/\theta-1/\nu)}}{(2\pi)^{d(1-\eta/\nu)}}\|F\widehat{x}(\cdot)\|^{2(1-\eta/\nu)}_{L_2(\mathbb{R}^d)}.$$

Hence

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{d^{\eta(1/\theta-1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}}\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu}(A+\varepsilon)^{\eta/\nu}.$$

Letting ε tend to zero we obtain (2.7).

Assume that there exists a constant

$$C < \frac{d^{\eta(1/\theta-1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}},$$

such that

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant C\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu}\left(\sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|_{L_2(\mathbb{R}^d)}^2\right)^{\eta/(2\nu)}.$$

Then

$$\sup_{\substack{\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}\leqslant\delta \\ \sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|_{L_2(\mathbb{R}^d)}^2\leqslant 1}} \|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant C\delta^{1-\eta/\nu} < \frac{d^{\eta(1/\theta-1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}}\delta^{1-\eta/\nu},$$

which contradicts (2.10).

From (2.7) for $\theta = 2$, $\nu \in \mathbb{N}$ and $\nu > \eta > 0$ we obtain the sharp inequality

$$\|(-\Delta)^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{d^{\eta(1-1/\nu)/2}}{(2\pi)^{d(1-\eta/\nu)/2}}\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu}\left(\sum_{j=1}^d \left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|_{L_2(\mathbb{R}^d)}^2\right)^{\eta/(2\nu)}.$$

Setting $\eta = 2$ we see that for all integers $\nu \geqslant 3$ we have the sharp inequality

$$\|\Delta x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{d^{1-1/\nu}}{(2\pi)^{d(1-2/\nu)/2}}\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-2/\nu}\left(\sum_{j=1}^d \left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|_{L_2(\mathbb{R}^d)}^2\right)^{1/\nu},$$

or (as $\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} = (2\pi)^{d/2}\|x(\cdot)\|_{L_2(\mathbb{R}^d)}$)

$$\|\Delta x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant d^{1-1/\nu}\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-2/\nu}\left(\sum_{j=1}^d \left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|_{L_2(\mathbb{R}^d)}^2\right)^{1/\nu}.$$

Now we deduce an analogue of Theorem 4 for $\Lambda = D^\alpha$.

Theorem 5. *Let $\nu > |\alpha| > 0$. Then the following sharp inequality holds*:

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}}\prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/(2\nu)}\|F\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-|\alpha|/\nu}\left(\sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|_{L_2(\mathbb{R}^d)}^2\right)^{|\alpha|/(2\nu)}. \tag{2.11}$$

Proof. Consider the extremal problem

$$\|D^\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \max, \qquad \|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant \delta^2, \quad \sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant 1. \tag{2.12}$$

For

$$0 < \varepsilon < \min\biggl\{\biggl(\frac{(2\pi)^d\alpha_j}{|\alpha|\delta^2}\biggr)^{1/(2\nu)} : \alpha_j > 0,\ j = 1, \dots, d\biggr\}$$

set

$$\widehat{\xi}_\varepsilon = \biggl(\frac{(2\pi)^d}{|\alpha|\delta^2}\biggr)^{1/(2\nu)} (\alpha_1^{1/(2\nu)}, \dots, \alpha_d^{1/(2\nu)}) - (\varepsilon_1, \dots, \varepsilon_d), \qquad \varepsilon_j = \begin{cases} \varepsilon, & \alpha_j > 0, \\ 0, & \alpha_j = 0, \end{cases}$$

and

$$B_\varepsilon = \{\xi \in \mathbb{R}^d \colon |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider the function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi) = \begin{cases} \dfrac{\delta}{\sqrt{\operatorname{mes} B_\varepsilon}} \biggl(1 + d\varepsilon^{2\nu} \dfrac{\delta^2}{(2\pi)^d}\biggr)^{-1/2}, & \xi \in B_\varepsilon, \\ 0, & \xi \notin B_\varepsilon. \end{cases}$$

Then

$$\|Fx_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)} = \delta^2 \biggl(1 + d\varepsilon^{2\nu} \frac{\delta^2}{(2\pi)^d}\biggr)^{-1} \leqslant \delta^2,$$

$$\begin{aligned} \sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|^2_{L_2(\mathbb{R}^d)} &= \frac{\delta^2}{(2\pi)^d \operatorname{mes} B_\varepsilon} \biggl(1 + d\varepsilon^{2\nu} \frac{\delta^2}{(2\pi)^d}\biggr)^{-1} \sum_{j=1}^d \int_{B_\varepsilon} |\xi_j|^{2\nu}\, d\xi \\ &\leqslant \frac{\delta^2}{(2\pi)^d \operatorname{mes} B_\varepsilon} \biggl(1 + d\varepsilon^{2\nu} \frac{\delta^2}{(2\pi)^d}\biggr)^{-1} \operatorname{mes} B_\varepsilon \biggl(\frac{(2\pi)^d}{|\alpha|\delta^2} \sum_{j=1}^d \alpha_j + d\varepsilon^{2\nu}\biggr) = 1. \end{aligned}$$

Therefore, $x_\varepsilon(\cdot)$ is an admissible function in problem (2.12). Let S denote the solution of this problem. Then

$$\begin{aligned} S &\geqslant \|D^\alpha x_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)} \\ &= \frac{\delta^2}{(2\pi)^d \operatorname{mes} B_\varepsilon} \biggl(1 + d\varepsilon^{2\nu} \frac{\delta^2}{(2\pi)^d}\biggr)^{-1} \int_{B_\varepsilon} |\xi_1|^{2\alpha_1} \cdots |\xi_d|^{2\alpha_d}\, d\xi \\ &= \frac{\delta^2}{(2\pi)^d} \biggl(1 + d\varepsilon^{2\nu} \frac{\delta^2}{(2\pi)^d}\biggr)^{-1} |\widetilde{\xi}_1|^{2\alpha_1} \cdots |\widetilde{\xi}_d|^{2\alpha_d}, \qquad (\widetilde{\xi}_1, \dots, \widetilde{\xi}_d) \in B_\varepsilon. \end{aligned}$$

Letting ε tend to zero we obtain

$$S \geqslant \biggl(\frac{\delta^2}{(2\pi)^d}\biggr)^{1-|\alpha|/\nu} |\alpha|^{-|\alpha|/\nu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/\nu}. \tag{2.13}$$

Set

$$\begin{aligned}
&\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)\\
&\quad= -\|D^\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \lambda_1\|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \lambda_2\bigg(\sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\bigg)\\
&\quad= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigg(-|\xi_1|^{2\alpha_1}\cdots|\xi_d|^{2\alpha_d} + (2\pi)^d\lambda_1 + \lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}\bigg)|Fx(\xi)|^2\,d\xi.
\end{aligned}$$

In the expression

$$-|\xi_1|^{2\alpha_1}\cdots|\xi_d|^{2\alpha_d} + (2\pi)^d\lambda_1 + \lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}$$

we make the substitution $|\xi_j|^2 = e^{t_j}$, $j = 1,\dots,d$. This yields the function

$$G(t) = -e^{(\alpha,t)} + (2\pi)^d\lambda_1 + \lambda_2\sum_{j=1}^d e^{\nu t_j} = e^{(\alpha,t)}H(t),$$

where $(\alpha,t) = \alpha_1 t_1 + \dots + \alpha_d t_d$ and

$$H(t) = -1 + (2\pi)^d\lambda_1 e^{-(\alpha,t)} + \lambda_2\sum_{j=1}^d e^{\nu t_j-(\alpha,t)}.$$

Set

$$\lambda_1 = \frac{1}{(2\pi)^d}\bigg(1-\frac{|\alpha|}{\nu}\bigg)\bigg(\frac{(2\pi)^d}{|\alpha|\delta^2}\bigg)^{|\alpha|/\nu}\prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/\nu}$$

and

$$\lambda_2 = \frac{1}{\nu}\bigg(\frac{(2\pi)^d}{|\alpha|\delta^2}\bigg)^{|\alpha|/\nu-1}\prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/\nu}.$$

The function $H(\cdot)$ is convex. It is easy to see that $H(\widehat{t}) = 0$, where

$$\widehat{t} = \frac{1}{\nu}\bigg(\ln\frac{(2\pi)^d\alpha_1}{|\alpha|\delta^2},\dots,\ln\frac{(2\pi)^d\alpha_d}{|\alpha|\delta^2}\bigg).$$

In addition, the gradient of $H(\cdot)$ vanishes at $\widehat{t}$. Hence $H(t) \geqslant 0$ for $t\in\mathbb{R}^d$, and therefore $G(t)\geqslant 0$ for $t\in\mathbb{R}^d$. Thus, $\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)\geqslant 0$.

For each function $x(\cdot)$ admissible in problem (2.12) we have

$$\begin{aligned}
-\|D^\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^d)} &\geqslant -\|D^\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \lambda_1\big(\|Fx(\cdot)\|^2_{L_2(\mathbb{R}^d)} - \delta^2\big)\\
&\qquad + \lambda_2\bigg(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)} - 1\bigg)\\
&= \mathcal{L}(x(\cdot),\lambda_1,\lambda_2) - \lambda_1\delta^2 - \lambda_2 \geqslant -\lambda_1\delta^2 - \lambda_2.
\end{aligned}$$

Hence

$$S \leqslant \lambda_1\delta^2 + \lambda_2 = \left(\frac{\delta^2}{(2\pi)^d}\right)^{1-|\alpha|/\nu} |\alpha|^{-|\alpha|/\nu} \prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/\nu}.$$

Taking (2.13) into account we obtain

$$S = \left(\frac{\delta^2}{(2\pi)^d}\right)^{1-|\alpha|/\nu} |\alpha|^{-|\alpha|/\nu} \prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/\nu}. \tag{2.14}$$

Let $x(\cdot) \in L_2(\mathbb{R}^d)$ and $D^{\nu e_j}x(\cdot) \in L_2(\mathbb{R}^d)$, $j = 1, \dots, d$, and set

$$A = \left(\sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{1/2}.$$

Then for $\varepsilon > 0$ и $\widehat{x}(\cdot) = x(\cdot)/(A+\varepsilon)$ we have

$$\sum_{j=1}^d \|D^{\nu e_j}\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}^d)} = \frac{A^2}{(A+\varepsilon)^2} < 1 \quad \text{and} \quad \|F\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)} = \frac{\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}}{A+\varepsilon}.$$

It follows from (2.14) that

$$\|D^\alpha\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant \frac{|\alpha|^{-|\alpha|/\nu}}{(2\pi)^{d(1-|\alpha|/\nu)}} \prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/\nu}\|F\widehat{x}(\cdot)\|^{2(1-|\alpha|/\nu)}_{L_2(\mathbb{R}^d)}.$$

Thus,

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}} \prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/(2\nu)}\|F\widehat{x}(\cdot)\|^{1-|\alpha|/\nu}_{L_2(\mathbb{R}^d)}(A+\varepsilon)^{|\alpha|/\nu}.$$

Letting ε tend to zero we obtain (2.11).

Suppose that there exists a constant

$$C < \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}} \prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/(2\nu)},$$

such that

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leqslant C\|F\widehat{x}(\cdot)\|^{1-|\alpha|/\nu}_{L_2(\mathbb{R}^d)}\left(\sum_{j=1}^d \|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{|\alpha|/(2\nu)}.$$

Then

$$\sup_{\substack{\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}\leqslant\delta\\ \sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\leqslant 1}} \|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant C\delta^{1-|\alpha|/\nu} < \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}}\prod_{\substack{j=1\\ \alpha_j\neq 0}}^d \alpha_j^{\alpha_j/(2\nu)}\delta^{1-|\alpha|/\nu}$$

in contradiction to (2.14).

Theorem 5 is proved.

2.2. Sharp inequalities in the metric of $L_\infty(\mathbb{R}^d)$. Set

$$\widehat{\gamma}_1=\frac{\nu-\eta-d/2}{\nu+d(1/2-1/p)},\qquad \widehat{q}_1=\frac{1}{1/2+\widehat{\gamma}_1(1/2-1/p)}$$

and

$$\widehat{C}_p(\nu,\eta)=\widehat{\gamma}_1^{-\widehat{\gamma}_1/p}(1-\widehat{\gamma}_1)^{-(1-\widehat{\gamma}_1)/2}\left(\frac{B(\widehat{q}_1\widehat{\gamma}_1/p+1,\widehat{q}_1(1-\widehat{\gamma}_1)/2)}{2|\nu-\eta-d/2|}\right)^{1/\widehat{q}_1}.$$

Theorem 8 in [7] has the following consequence.

Theorem 6. *Let $1\leqslant p\leqslant\infty$ and $\widehat{\gamma}_1\in(0,1)$. Assume that*

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}_1}(\omega)}{\big(\sum_{j=1}^n\widetilde{\varphi}_j^2(\omega)\big)^{\widehat{q}_1(1-\widehat{\gamma}_1)/2}}J(\omega)\,d\omega<\infty.$$

Then the following sharp inequality holds:

$$\|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)}\leqslant\frac{\widehat{C}_p(\nu,\eta)I^{1/\widehat{q}_1}}{(2\pi)^{d(1+\widehat{\gamma}_1)/2}}\|Fx(\cdot)\|^{\widehat{\gamma}_1}_{L_p(\mathbb{R}^d)}\left(\sum_{j=1}^n\|D_jx(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{(1-\widehat{\gamma}_1)/2}. \tag{2.15}$$

Let $\Lambda=\Lambda_\theta^{\eta/2}$ and $D_j=D^{\nu e_j}$, $j=1,\dots,d$. Then the quantity I in Theorem 6 has the form

$$I=\int_{\Pi^{d-1}}\frac{\big(\sum_{k=1}^d\widetilde{t}_k^\theta(\omega)\big)^{\widehat{q}_1\eta/\theta}J(\omega)\,d\omega}{\big(\sum_{k=1}^d\widetilde{t}_k^{2\nu}(\omega)\big)^{\widehat{q}_1(1-\widehat{\gamma}_1)/2}}. \tag{2.16}$$

It follows from (2.4) and (2.5) that $I<\infty$. Then Theorem 6 has the following corollary.

Corollary 4. *Let $1\leqslant p\leqslant\infty$ and $\nu-d/2>\eta>0$. Then the sharp inequality*

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_\infty(\mathbb{R}^d)}\leqslant\frac{\widehat{C}_p(\nu,\eta)I^{1/\widehat{q}_1}}{(2\pi)^{d(1+\widehat{\gamma}_1)/2}}\|Fx(\cdot)\|^{\widehat{\gamma}_1}_{L_p(\mathbb{R}^d)}\left(\sum_{j=1}^d\|D^{\nu e_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{(1-\widehat{\gamma}_1)/2}$$

holds, where I is defined by (2.16).

If $\Lambda = D^\alpha$ and $D_j = D^{\nu e_j}$, $j = 1, \dots, d$, then for I in Theorem 6 we have

$$I = \int_{\Pi^{d-1}} \frac{\bigl(\widetilde{t}_1^{\alpha_1}(\omega) \cdots \widetilde{t}_d^{\alpha_d}(\omega)\bigr)^{\widetilde{q}_1} J(\omega)\, d\omega}{\bigl(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\bigr)^{\widetilde{q}_1(1-\widetilde{\gamma}_1)/2}}, \tag{2.17}$$

where

$$\widetilde{\gamma}_1 = \frac{\nu - |\alpha| - d/2}{\nu + d(1/2 - 1/p)} \quad \text{and} \quad \widetilde{q}_1 = \frac{1}{1/2 + \widehat{\gamma}_1(1/2 - 1/p)}.$$

From (2.4) and (2.5) we obtain $I < \infty$. Then Theorem 6 yields the following result.

Corollary 5. *Let* $1 \leqslant p \leqslant \infty$ *and* $\nu - d/2 > |\alpha| > 0$. *Then the sharp inequality*

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leqslant \frac{K_p(\nu, |\alpha|) I^{1/\widetilde{q}_1}}{(2\pi)^{d(1+\widetilde{\gamma}_1)/2}} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widetilde{\gamma}_1} \biggl(\sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2\biggr)^{(1-\widetilde{\gamma}_1)/2}$$

holds, where

$$K_p(\nu, |\alpha|) = \widetilde{\gamma}_1^{-\widetilde{\gamma}_1/p} (1 - \widetilde{\gamma}_1)^{-(1-\widetilde{\gamma}_1)/2} \biggl(\frac{B(\widetilde{q}_1 \widetilde{\gamma}_1/p + 1, \widetilde{q}_1(1 - \widetilde{\gamma}_1)/2)}{2(\nu - |\alpha| - d/2)}\biggr)^{1/\widetilde{q}_1}$$

and I *is defined by* (2.17).

Bibliography

[1] F. Carlson, "Une inégalité", *Ark. Mat. Astron. Fys. B* **25** (1935), 1, 5 pp.

[2] V. I. Levin, "Sharp constants in Carlson-type inequalities", *Dokl. Akad. Nauk SSSR* **59**:4 (1948), 635–638. (Russian)

[3] F. I. Andrianov, "Multidimensional analogues of Carlson's inequality and of its generalizations.", *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1967, no. 1, 3–7. (Russian)

[4] S. Barza, V. Burenkov, J. Pečarić and L.-E. Persson, "Sharp multidimensional multiplicative inequalities for weighted L_p spaces with homogeneous weights", *Math. Inequal. Appl.* **1**:1 (1998), 53–67.

[5] K. Yu. Osipenko, "Optimal recovery of operators and multidimensional Carlson type inequalities", *J. Complexity* **32**:1 (2016), 53–73.

[6] K. Yu. Osipenko, "Inequalities for derivatives with the Fourier transform", *Appl. Comput. Harmon. Anal.* **53** (2021), 132–150.

[7] K. Yu. Osipenko, "Optimal recovery and generalized Carlson inequality for weights with symmetry properties", *J. Complexity* **81** (2024), 101807, 35 pp.

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