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Main Recent Scientific Results of the Chair of General Problems of Control

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Abstract—In this paper, we survey recent results obtained by the staff of the Chair of General Problems of Control.

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1. INTRODUCTION

The scientific interests of the members of the Chair of General Problems of Control (GPC) are broad and various. They are related to the problems of the general theory of extremum, control theory, approximation theory, problems of fluid flow about a bounded body, problems of optimal recovery, and many others. Several questions concerning this topics are discussed in the current survey. The below text consists of the sections each of which is written by one of the employees of the Chair of GPC. The sections are written in the alphabet order of the surnames of their authors.

2. KOLMOGOROV WIDTHS

The section is prepared by A.A. Vasilieva. Let X be a normed space, $M \subset X$, $n \in \mathbb{Z}_+$, $\mathcal{L}_n(X)$ be the union of all subspaces in X of dimension no higher than n. The Kolmogorov n-width of the set M in the space X is the value

$$d_n(M,\, X) = \inf_{L\in\mathcal{L}_n(X)} \sup_{x\in M} \inf_{y\in L} ||x-y||.$$

Let $m,\, k \in \mathbb{N}$, $1 \leqslant p < \infty$, $1 \leqslant \theta < \infty$. By $l^{m,k}_{p,\theta}$ we denote the space $\mathbb{R}^{mk}$ with the norm $||(x_{i,j})_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant k}||_{l^{m,k}_{p,\theta}} = \left(\sum_{j=1}^k \left(\sum_{i=1}^m |x_{i,j}|^p\right)^{\theta/p}\right)^{1/\theta}$. For $p=\infty$ or $\theta=\infty$ the definition is modified naturally. By $B^{m,k}_{p,\theta}$ we denote a unit ball of the space $l^{m,k}_{p,\theta}$. In the case $k=1$ the space and unit ball are denoted, respectively, by l^m_p and B^m_p.

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Order estimates of the values $d_n\left(B_p^N,\, l_q^N\right)$ up to multiplicative constants depending only on q are known for $1 \leqslant q < \infty$ and arbitrary p, and also for $q = \infty$ and $p \geqslant 2$ (for $p \geqslant q$ and $p = 1$, $q = 2$ the exact values are computed) (see the history of the question and the bibliography, e.g., in [1]).

E.M. Galeev [2] obtained order estimates $d_n\left(\cap_{\alpha\in A}\nu_\alpha B_{p_\alpha}^N,\, l_q^N\right)$ at $N = 2n$ (here, $\nu_\alpha > 0$, $\alpha \in A$). This result is extended by A.A. Vasilieva to the case $N \geqslant 2n$. It is shown that the estimate of the width of the intersection of balls reduces to computation of the infimum of the set of values of the form $\nu d_n\left(B_p^N, l_q^N\right)$. The exact formulation of the result is given in [3] and [4, Proposition 1]; note that Proposition 1 from [4] can easily be extended to the case of an arbitrary A, reasoning in the same way as in [5, Section 5]. The obtained result allows extending the Galeev theorem [6] about the estimates of the widths of a finite intersection of Sobolev classes on a one-dimensional torus to the case of small smoothness for $q > 2$ (see [4]).

In addition to that, the order estimates $d_n\left(\cap_{\alpha\in A}\nu_\alpha B_{p_\alpha,\theta_\alpha}^{m,k},\, l_{q,\sigma}^{m,k}\right)$ are obtained for $2 \leqslant q$, $\sigma < \infty$, $n \leqslant \frac{mk}{2}$; the denotations and formulation of the result are given in [5].

3. ON THE PROBLEM OF STREAMLINE

A.V. Gorshkov investigated the two-dimensional problem of the flow of an incompressible fluid about a bounded body with no-slip condition at the boundary, the dynamics of which is described by the nonlinear vortex equation (the Helmholtz equation). The main problem was to construct the no-slip boundary condition in the vortex form and to prove the solvability in the external domain $\Omega \subset \mathbb{R}^2$.

The initial-boundary value problem with the no-slip condition at the domain boundary in the vortex representation transits to the integral relations given on the entire exterior of the bounded body. These relations are the orthogonality conditions of the rotor function to harmonic functions. For the exterior of a simply connected domain with a given horizontal flow at infinity $\mathbf{v}_\infty = (v_\infty, 0)$, these conditions on the vortex function w are given by

$$\frac{1}{2\pi}\int\limits_{\Omega}\frac{w(x)}{\Phi(z)^k}dx = \begin{cases}0, & k \in \mathbb{N}\cup\{0\}, \quad k \neq 1;\\ iv_\infty, & k = 1,\end{cases} \tag{1}$$

where $\Phi(z)$ is the Riemann mapping from Ω to the exterior of the disk given by $\Phi(z) = z + O\left(\frac{1}{z}\right)$.

The problem of recovering a solenoidal vector field $\mathbf{v}$ by a vortex function w (in the English literature it is called the div-curl problem) has the unique problem up to harmonic fields. These fields are circulation and have an infinite kinetic energy. Because the orthogonality condition (1) at $k = 0$ implies that the average of the rotor is zero, according to the Stokes formula, under the no-slip condition the flow has no circulation at infinity. This fact allows obtaining mean-squared estimates for the vector field. The unique solvability of the div-curl problem with the no-slip condition is proved along with the estimate $||\mathbf{v}(\cdot) - \mathbf{v}_\infty||_{H^1(\Omega)} \leq C||w(\mathbf{x})(1 + |\mathbf{x}|^2)^{N/2}||_{L_2(\Omega)}$, valid for any $N > 1$. For linear vortex equations, the orthogonality conditions (1), distributed over the entire infinity domain Ω, can be reduced to boundary conditions, distributed now only over the boundary of the domain $\partial\Omega$. For the exterior of the disk of radius r_0, the no-slip boundary condition in terms of the Fourier coefficients $w_k(t, \cdot)$ of the vortex function $w(t, \cdot)$ is given by

$$r_0\frac{\partial w_k(t, r)}{\partial r}\Big|_{r=r_0} + |k|w_k(t, r_0) = 0, \quad k \in \mathbb{Z}.$$

The linear operator of the vortex equation with such boundary condition generates the degenerate Fourier-like transform F, which has a homogeneous kernel with the basis function e_0 and for which the Parseval equality becomes $||f||^2 = ||F[f]||^2 + (f, e_0)^2,\ F[e_0] = 0$.

Using these boundary conditions, Gorshkov [7] found an explicit formula of solution to the Stokes system of the flow about a circular cylinder in the vortex form. In work [8] he constructed a similar boundary condition for the Navier–Stokes system. The transform F itself was studied in [9].

4. OPTIMIZATION OF TRAJECTORIES OF A SPACE VEHICLE. CREATION AND OPTIMIZATION OF MODERN FINANCIAL INSTRUMENTS AND TECHNOLOGIES

M.P. Zapletin studies complex problems of trajectory optimization requiring for their solution a synthesis of the methods of local and multiextremal optimization, optimal control, space dynamics, mechanics of space flight, celestial mechanics, and numerical methods. He formulated a three-dimensional space dynamics problem of through optimization of the trajectory of interplanetary flight of a spacecraft with a single functional, detailed consideration of planet-centric segments without the use of gravispheres of zero elongation, with a combined thrust and phasing. He proposed a method for solving multiextremal optimization problems for trajectories of interplanetary flights with return to the Earth taking into account ephemerides, strict phasing, bounded combined large and small piecewise-continuous thrust, including the solution to series auxiliary problems in a simplified formulation and parameter continuation of the solution. The numerical methods for solving boundary value problems of the maximum principle arising in the control of the combination of dynamic systems were developed taking into account the effect of accuracy loss and restructuring of the trajectory when the number of active segments during parameter continuation of the solution varies. The results of numerical solution of the problems were described in [10, 11].

M.P. Zapletin also considers several optimization problems for constructing the orbit of a satellite of Earth's remote sensing (ERS), estimating the schedule and the capabilities for imaging the region of interest on the Earth's surface. He provided a program of visualization of the orbit of any available commercial ERS spacecraft at the required time period, estimation, and planning of the given territory by a certain spacecraft. The computational part of the problem is based on the SGP4 model using publicly available TLE data for ERS satellites, on formulas of spherical trigonometry, and on heuristic methods of computation reduction [12].

M.P. Zapletin presented studies associated with elaboration of the concept of duality of commodities in part of development of its theoretical aspects and mathematical apparatus with development and generalization of the mathematical toolkit which allows processing time series for the purpose of revealing the property of duality of a commodity with respect to a chosen commodity (in particular, to gold), as well as with development of schemes, algorithms, and decision making rules in the question of estimating the duality of commodities with respect to a given commodity. The results of the study allow forming a financial instrument that is the basis for cash collateral. The complex of the obtained scientific results can be used in creating an investment reserve contour embedded into the existing monetary and financial system without conflicts with international commitments and regulations, as well as in constructing clearance and settlement systems of international unions [13–17]. The monographs *Dual Products* [13] was awarded the Prize The Best Economic Book of 2023 in the category "Monographs. Economic Studies" by the International Union of Economists (IUE).

5. REGULARITY OF SUB-RIEMANNIAN GEODESICS

The authors of the section are L.V. Lokucievskii and M.I. Zelikin. A growing interest of mathematicians to nonholonomic and sub-Riemannian problems arose starting from the middle of the 20th century in view of works of prominent mathematicians such as É. Cartan, M.L. Gromov, P. Montgomery, J. Michell, A.M. Vershik, and A.A. Agrachev. Many applied problems are naturally formulated as problems of searching sub-Riemannian geodesics: problems of kinematic control, problems of quantum system control, problems of image processing, and others. The most secret and intriguing object in this theory is abnormal geodesics arising as critical points of an exponential mapping (end-point map). Since the beginning of the interest to special geodesics, there appeared a question about the methods for their study and, in particular, the question about their smoothness is not finally resolved by now. It not only has an important theoretical significance, but also is of great practical interest.

In 2023, M.I. Zelikin and L.V. Lokucievskii obtained the first result about regularity of sub-Riemannian geodesics not using any a priori assumptions [18]. They proved that the velocity on any sub-Riemannian geodesics must be L_p-Hölder. Note that the results of other authors in this field always relied upon particular very limiting a priori assumptions about the structure of the sub-Riemannian manifold or the geodesic itself.

The smoothness of geodesics in Riemannian geometry is simple to study: any Riemannian geodesic satisfies the Euler–Lagrange system of equations and therefore must be smooth (or even analytic).

Sub-Riemannian geodesics do not have to obey the Euler–Lagrange equation; instead of it they obey a Hamiltonian system of inclusions of Pontryagin's maximum principle. If the principal part of the Hamiltonian is not degenerate, then the corresponding geodesic is normal, it satisfies the Euler–Lagrange equations, and hence must be smooth. However, when we attempt to study the smoothness of abnormal geodesics, the principal part of Pontryagin's Hamiltonian degenerates and the differential inclusion is not equivalent to any ordinary differential equation. Some exception would be sub-Riemannian manifolds of depth $s = 2$, because in this case, according to the Goh condition, abnormal geodesics are absent (see [19]). However, even in the case $s = 3$ studying the smoothness of abnormal geodesics in this way is very difficult (however, in this direction there are several results [20]). At the depth $s = 4$ there are almost no general results (work [21] was the only exception as of beginning of 2023).

In 2016, work [22] was published, in which Hakavuori and Le Donne proved a remarkable result: sub-Riemannian abnormal geodesics cannot have corners. It is typical that the proof of this result does not rely upon Pontryagin's maximum principle, but is obtained using rather different considerations. Formally, this result does not control the smoothness of geodesics. Nevertheless, it has many interesting consequences. For instance, in work [23] Belotto da Silva et al. proved that geodesics on three-dimensional manifolds, independently of the depth, must be C^1-smooth. The above mentioned work [21] also relies upon the result about the absence of corners.

Nevertheless, up to now, despite continuing efforts of many leading mathematicians, it is not proved that abnormal geodesics on sub-Riemannian manifolds in the general case are smooth. We have widely known examples [19, Subsection 12.6.1] of abnormal extremals (but not geodesics) on sub-Riemannian manifolds which are not smooth, but we still know no examples of nonsmooth abnormal geodesic.

In the work [18] of L.V. Lokucievskii and M.I. Zelikin used a certain nontrivial dual interpolating estimate for the abnormal control to obtain the following result.

Theorem . *On any sub-Riemannian manifold of constant rank, any geodesic has an L_p-Hölder derivative for any $1 \leqslant p < \infty$.*

Thus, if nonsmooth abnormal geodesics exist, they lie in a very narrow class of curves the velocity on which is L_p-Hölder with a certain exponent $0 < \alpha \leqslant 1$, but is not smooth. For $\alpha > \frac{1}{p}$ the fact is that this class is empty. Before 2023, there were no attempts to construct a nonsmooth sub-Riemannian geodesic exactly in this class. In work [18], Lokucievskii and Zelikin provided a theorem, which, either is formulated bulkier, but provides a qualitative uniform estimate of the exponent α on any compact directly related with the depth s of the sub-Riemannian manifold.

This result has not only a clear theoretical significance, but also can be used to justify numerical methods for finding solutions, namely, in work [18] Lokucievskii and Zelikin obtained the two important practical consequences: they (1) proved the rapid decrease rate in the Fourier coefficients of the geodesics and (2) proved the polynomial efficiency of approximation of the velocity on geodesics by C^1-smooth curves (for instance, by splines).

One of the most important problems of quantum computations is associated with construction of minimal quantum chains—short sequences of quantum gates realizing with some accuracy a given unitary operator on a system of n qubits (in the group $SU(2^n)$). M. Nelson and M. Dowling (2006) proved that decomposition of a unitary operator into a sequence of gates can be equivalently considered determination of the shortest path in $SU(2^n)$ connecting an identity operator given in a specially Riemannian metric on $SU(2^n)$ containing some penalty parameter $p > 0$ relating the path length and the complexity of quantum chain. The penalty parameter must be very large in order to neutralize the uncontrolled part of error. As $p \to \infty$, the mentioned considerations lead to a sub-Riemannian problem on the group $SU(2^n)$, geodesics in which can be found numerically and yield a decomposition of this unitary operator into a sequence of quantum gates. The quantitative estimates to the Hölder exponents of sub-Riemannian geodesics obtained in the work of M.I. Zelikin and L.V. Lokucievskii allow estimating the efficiency of numerical algorithms in this problem. In addition to that, the sub-Riemannian geometry and the peculiarities of the sub-Riemannian metric are tightly related with the optimal solutions in problems of control of closed quantum systems.

6. LOCAL CONTROLLABILITY AND OPTIMALITY

The section is prepared by G.G. Magaril-Ilyaev. The theory of optimal control—is the most important component of the general theory of extremum, and in applied question it is one of the most needed theories. For the problem of optimal control, the concept of the local infimum trajectory—a function generalizing the notion of optimal trajectory—is introduced. This is a function at which the objective functional reaches a local minimum on the closure of the set of admissible trajectories considered as a subset of continuous functions. The local infimum trajectory is not, generally speaking, an admissible trajectory, but is, obviously, a uniform limit of such trajectories. The optimal trajectory can be not existing, but the existence of a local infimum trajectory is completely sufficient for applications. For a local infimum trajectory the necessary conditions of the first and second orders were obtained [24–27]. If, in particular, a local infimum trajectory is an optimal trajectory, then the obtained conditions contain classical necessary conditions of the first order (Pontryagin's maximum principle) the known optimality conditions of the second order, as well as other relations, which, as shown by examples, provide an additional and very substantial information about the optimal process. In this sense the obtained statements strengthen the known results.

The notion of controllability of a control system is one of the most important in the theory of optimal control. The notion of controllability by a system of ordinary differential equations with generic boundary conditions is introduced, and the conditions are derived that guarantee the controllability not only for the original control system, but also for systems close to it, and, furthermore, for the controllability of close systems it suffices to have just continuity of the mappings in its definition [28–31]. In practice, close mappings arise as a consequence of inaccuracy of prescribing initial data and/or as approximation of "complex" mappings by simpler ones, which, as a rule, are just continuous. This issues are tightly with the questions about the continuous dependence of a solution to a differential equation on its right-hand side and boundary conditions. A general statement about the continuous dependence of the solution on the right-hand side and generic boundary conditions was proved, which leads to several well-known results [32].

It is worth noting that the proof of many of the above mentioned statements required development of novel mathematical tools, in particular, theorems about existence of an implicit function not only in the original mapping, but also in the mappings close (in a certain sense) to the original one [29]. All the above mentioned studies were performed together with E.R. Avakov.

7. ON SOME MULTIDIMENSIONAL STRICT INEQUALITIES OF KOLMOGOROV TYPE

The section was prepared by K.Yu. Osipenko. The Kolmogorov-type inequalities for derivatives on a straight line are traditionally understood as the inequalities of the form

$$\left|\left|x^{(k)}(\cdot)\right|\right|_{L_q(\mathbb{R})} \leqslant K\,||x(\cdot)||^{\alpha}_{L_p(\mathbb{R})}\left|\left|x^{(n)}(\cdot)\right|\right|^{\beta}_{L_r(\mathbb{R})}, \tag{2}$$

where $0 \leqslant k < n$ are integers, $1 \leqslant p, q, r \leqslant \infty$, $\alpha, \beta \geqslant 0$. In 1939, A.N. Kolmogorov found exact constants in (2) at $p = q = r = \infty$ in the general case, that is, for any $n \geq 2$ and $0 < k < n$. This result is the most striking in this topic. The results similar in their completeness to the Kolmogorov ones were obtained on a straight line only in the three cases ($p = q = r = 2$—G.H. Hardy, J.E. Littlewood, and G. Pólya (1934), $p = q = r = 1$—E. Stein (1957), $p = r = 2$, $q = \infty$—L.S. Taikov (1968)).

If we replace the function $x(\cdot)$ with its Fourier transform, then at $r = 2$ and $q = \infty, 2$ we can obtain (see [33]) even more general inequalities valid not only for all $n > k$ ($n > k + d/2$ in the case $q = \infty$), but also for all $1 \leqslant p \leqslant \infty$ ($2 < p \leq \infty$ in the case $q = 2$). We put $\gamma = \frac{n-k-d/2}{n+d(1/2-1/p)}$, $\widetilde{q} = \frac{1}{1/2+\gamma(1/2-1/p)}$.

Theorem. *Let $1 \leqslant p \leqslant \infty$, $k \geqslant 0$, $k + p > 1$, and $n > k + d/2$. Then the strict inequality holds:*

$$\left|\left|(-\Delta)^{k/2}x(\cdot)\right|\right|_{L_\infty(\mathbb{R}^d)} \leqslant K_p(k,n)||Fx(\cdot)||^{\gamma}_{L_p(\mathbb{R}^d)}\left|\left|(-\Delta)^{n/2}x(\cdot)\right|\right|^{1-\gamma}_{L_2(\mathbb{R}^d)},$$

where

$$K_p(k,n) = \frac{\gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{2}}}{(2\pi)^{d\frac{2n-k-d/p}{2n+d(1-2/p)}}}\left(\frac{B\left(\widetilde{q}\gamma/2+1, \widetilde{q}(1-\gamma)/2\right)\pi^{d/2}}{(n-k-d/2)\Gamma(d/2)}\right)^{1/\widetilde{q}},$$

$Fx(\cdot)$ is the Fourier transform of the function $x(\cdot)$, and $B(\cdot,\cdot)$ is the Euler B-function.

We put

$$\widetilde{\gamma}=\frac{n-k}{n+d(1/2-1/p)},\quad q_1=\frac{1}{\widetilde{\gamma}(1/2-1/p)},$$

$$\widetilde{K}_p(k,n)=\frac{\widetilde{\gamma}^{-\frac{\widetilde{\gamma}}{p}}(1-\widetilde{\gamma})^{-\frac{1-\widetilde{\gamma}}{2}}}{(2\pi)^{d\widetilde{\gamma}/2}}\left(\frac{B\left(q_1\widetilde{\gamma}/2+1,q_1(1-\widetilde{\gamma})/2\right)\pi^{d/2}}{(n-k)\Gamma(d/2)}\right)^{1/q_1}.$$

Theorem. *Let $k\geqslant 0$, $n>k$, and $2<p\leqslant\infty$. Then the strict inequality holds*:

$$\left|\left|(-\Delta)^{k/2}x(\cdot)\right|\right|_{L_2(\mathbb{R}^d)}\leqslant\widetilde{K}_p(k,n)||Fx(\cdot)||^{\widetilde{\gamma}}_{L_p(\mathbb{R}^d)}\left|\left|(-\Delta)^{n/2}x(\cdot)\right|\right|^{1-\widetilde{\gamma}}_{L_2(\mathbb{R}^d)}.$$

8. CONTRACTED DIMENSIONS, RECOVERY OF RIDGE FUNCTIONS, POLYNOMIAL APPROXIMATIONS

K.S. Ryutin obtained [34, 35] the results on the problem of integer-value contracted dimensions proposed by L. Fukshansky, D. Needell, and B. Sudakov in 2019. By a vector Ax, where A is an integer-value $(m\times d)$-matrix (of measurements), we need to recover the vector x, which is assumed *s-sparse*, that is, its carrier has a cardinality s, $s<d$. In [36], S.V. Konyagin and B. Sudakov provided a structure of a well measurement matrix with elements small in absolute values, that is, $(m\times d)$-matrices $A, m<d$, such that any s-sparse vector $x\in\mathbb{Z}^d$, $s<d/2$, can be uniquely recovered by the vector Ax. K.S. Ryutin proposed a recovering algorithm for this matrix and estimated its complexity (that is, the number of operations). As a consequence, for a large N and any $s\leqslant c_1N/\log N$ there exists a Boolean $(c_2s\log N)\times N$-matrix with an efficient unique recovery of any s-sparse vectors from $\mathbb{Z}^N$.

T.I. Zaitseva, Yu.V. Malykhin, and K.S. Ryutin (see [37]) proposed an algorithm for recovering a ridge function ("plane wave") by its values at a finite number of points the number of which depends polynomially on the dimension n if the generating function φ belongs to the natural class of analytic functions. That is, we recover a function $f(x)=\varphi(\langle a,x\rangle)$, where $a,x\in\mathbb{R}^n$, $|a|=1$, $\langle\cdot,\cdot\rangle$ is a scalar product, and φ is an analytic function in the neighborhood of the segment $[-1,1]$, by a set of values (data with uncertainty) of the function f at the points $x_1,\dots,x_N$ from the unit ball $\mathbb{R}^n$; in this case a and φ are unknown to us. An original (probabilistic) algorithm was proposed that combines approaches from the mathematical statistics and theory of extrapolation of polynomials of complex variable, and its accuracy is estimated. This work substantially supplements the results of several authors on this topic.

Yu.V. Malykhin and K.S. Ryutin in [38] constructed explicit constructive polynomial high-accuracy approximations of locally constant functions on a union of a finite number of disjunct segments on a straight line. The obtained upper bounds of the approximation accuracy correctly reflect the dependence on the geometric characteristics of the family of segments in the two most interesting asymptotic modes (when all segments are sufficiently small and when some of them turn out to be very close). The formulation refers to the classical theory of polynomial approximation and is associated with the names of E.I. Zolotarev, N.I. Akhiezer, S.N. Bernshtein, G. Sege, G. Faber, but is also useful to study discrete objects (the complexity of Boolean functions and tensors). Of special interest is the so called method of approximation improvement applied in the theory of complexity and theory of widths.

9. ON THE PROBLEM OF STABILIZATION OF SOME SYSTEMS OF HYDRODYNAMIC TYPE BY MEANS OF FEEDBACK CONTROL

The described approach was presented in the work of A.V. Fursikov [39], in which he also provided a detailed reference list on this topic. The main hydrodynamic system for which the stabilization problem is studied here is the system of Navier–Stokes equations of an incompressible fluid given in a bounded domain $\Omega\subset\mathbb{R}^3$. The unknown functions in it are the velocity and pressure of fluid $(v(t,x),p(t,x),\ (t,x)\in\mathbb{R}_+\times\Omega)$, as well as the control $u(t,x),\ (t,x)\in\mathbb{R}_+\times\partial\Omega$, given at the domain boundary and coinciding with the boundary value of the fluid velocity. The given functions are the initial velocity value $v_0(x) = v(t,x)|_{t=0}$, the external force $f(x)$, $x \in \Omega$, and the stationary solution $\hat{v}(x)$, $x \in \Omega$, to the Navier–Stokes system with the same right-hand side $f(x)$.

The stabilization problem consists in constructing a control $u(t,x)$ such that the velocity of fluid $v(t,x)$ obtained with it satisfied the condition $||v(t,\cdot) - \hat{v}(\cdot)|| \to 0$ as $t \to \infty$, where $||\cdot||$ is the corresponding norm. This problem was completely solved under an additional condition that the value $||\hat{v} - v_0||$ is sufficiently small (that is, under the locality condition). In the nonlocal case, that is, when this condition is not fulfilled, just the first step towards the complete solution of the problem is made that contains in the following. First of all, by analogy with the formulation of the millenium problem about the existence of a smooth solution to the three-dimensional Navier–Stokes problem, we begin with the case when the external force $f(x)$ is zero. As is well-known from the local stabilization theory, in this case the problem can be reduced to a boundary value problem with periodic boundary conditions (that is, replace the domain Ω with the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$) and a control given in the right-hand side of the equation and concentrated in a certain special subdomain of the torus $\omega \subset \mathbb{T}^3$. On the one side, the nonlocal stabilization theory needs to be constructed in the class of sufficiently smooth functions, where the solution to the boundary value problem is unique, and, on the other side, as the phase space, it is useful to take the space $(L_2(\mathbb{T}^3))^3$, in which it is easier to trace the flow dynamics. To satisfy both conditions, we proceed from the stabilization problem for the Navier–Stokes system to a similar problem for the Helmholtz system, whose solution $w(t,x)$ is associated with the fluid velocity by the relation $w(t,x) = \mathrm{curl} v(t,x)$ (that is, w is the vortex of the velocity v). It is well-known that a nonlinear operator in the Helmholtz system is given by $B(w) = \Phi(w)w + B_\tau(w)$, where $\Phi(w)$ is a functional and $B_\tau(w)$ and w are orthogonal in the space $(L_2(\mathbb{T}^3))^3$. This easily implies that the main difficulties in constructing the stabilization for the Helmholtz system are related to the first term of the nonlinear operator $B(w)$. Hence, the following solution scheme of the stabilization problem was accepted. At the first stage, the operator $B(w)$ is replaced with $\Phi(w)w$, and the stabilization for the obtained system is constructed. This problem has been successfully solved. At the second stage, we should return to the original operator $B(w)$ and solve the problem. Unfortunately, the second step is made just for a benchmark example, in which the Burgers equation is taken instead of the Navier–Stokes system.

The limited volume of the paper did not allow us to speak in details about other interesting works of the members of the chair such as the studies of Corresponding Member of the Russian Academy of Sciences V.Yu. Protasov on harmonic analysis, stability of dynamic systems, self-similar sets, Professor A.S. Demidov on explicit numerically realizable formulas for the Poincaré–Steklov operators and solutions to some other equations of the mathematical physics, Associate Professor A.S. Kochurov (together with A.S. Demidov) on approximate computation of the nth derivative by the measurement data of a function and (together with Professor V.M. Tikhomirov) about extrapolation of polynomials with real coefficients to the complex plane.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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