OPTIMAL RECOVERY OF THE WAVE EQUATION SOLUTION BY INACCURATE INPUT DATA

K. YU. OSIPENKO, N. D. VYSK

ABSTRACT. In this paper the problem of optimal recovery of the wave equation solution by inaccurate values of Fourier coefficients of a function defined the initial form of the string is considered. A solution of more general problem of recovery of operator defined on a weighted space of vectors from l_2 by inaccurate values of its coordinates is given.

1. Statement of the problem

Consider the wave equation with zero boundary conditions and the initial velocity equals zero

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0. \end{aligned} \tag{1}$$

It is known that the precise solution of this problem has the following form

$$u(x,t) = \sum_{j=1}^{\infty} a_j(f)\cos jt \sin jx, \tag{2}$$

where

$$a_j(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin jx\,dx$$

are the Fourier coefficients of $f(x)$.

Assume that $f(\cdot) \in W_2^n([0,\pi])$, where

$$W_2^n([0,\pi]) = \{\, f(\cdot) \in L_2([0,\pi]) : f^{(n-1)}(\cdot) \text{ abs. cont. on } [0,\pi], \\ \|f^{(n)}(\cdot)\|_{L_2([0,\pi])} \le 1 \,\},$$

The research was carried out with the financial support of the Russian Foundation for Basic Research (grant nos. 05-01-00275, 06-01-81004, 05-01-00261, and 06-01-81004).

and

$$\|g(\cdot)\|_{L_2([0,\pi])}=\sqrt{\frac{2}{\pi}\int_0^\pi |g(x)|^2\,dx}.$$

We assume that we know inaccurate values of the first N Fourier coefficients of $f(\cdot)$, $y_1,\dots,y_N$, moreover

$$\sum_{j=1}^N |a_j(f)-y_j|^2\le\delta^2,\quad \delta>0. \tag{3}$$

We state the problem of finding of an optimal recovery method for the solution of problem (1) at the time T on the class $W_2^n([0,\pi])$ by the information operator F_δ^N which associates with every function $f(\cdot)\in W_2^n([0,\pi])$ a vector $y=(y_1,\dots,y_N)$ satisfying condition (3).

Any operators $\varphi\colon\mathbb{R}^N\to L_2([0,\pi])$ are admitted as recovery methods. The quantity

$$e(T,W_2^n([0,\pi]),F_\delta^N,\varphi) \\ =\sup_{\substack{f(\cdot)\in W_2^n([0,\pi]),\ y=(y_1,\dots,y)\in\mathbb{R}^N\\ \sum_{j=1}^N |a_j(f)-y_j|^2\le\delta^2}} \|u(\cdot,T)-\varphi(y)(\cdot)\|_{L_2([0,\pi])}$$

is called the *error* of the method φ. The quantity

$$E(T,W_2^n([0,\pi]),F_\delta^N)=\inf_{\varphi\colon\mathbb{R}^N\to L_2([0,\pi])} e(T,W_2^n([0,\pi]),F_\delta^N,\varphi)$$

is called the *error of optimal recovery*, and a method delivering the lower bound is called *optimal recovery method*.

The idea of solution of the stated problem is based on the method of optimal recovery of linear operators developed in the papers [1] and [2] (see also [3]). The essential part of this method is reduction of the original problem to a minimization problem with constraints which in the present case is reducing to a problem of linear programming which can be exactly solved by the Lagrange principle of constraints removing.

Now consider a more general problem of optimal recovery to which the stated problem can be reduced. Let the operator $Q\colon X\to l_2$ be defined by the equality

$$Qx=(\eta_1x_1,\eta_2x_2,\dots),\ j\in\mathbb{N},$$

where $x=(x_1,x_2,\dots)\in X$, and

$$X=\left\{x=(x_1,x_2,\dots):\|x\|_X=\left(\sum_{j=1}^\infty \nu_j|x_j|^2\right)^{1/2}<\infty\right\},$$

$\nu_j > 0$, $j \in \mathbb{N}$. Set $\mu_j = \eta_j^2$ and assume that $\mu_j/\nu_j \to 0$ as $j \to \infty$. Then for all $x \in X$, $Qx \in l_2$. We are interested in the problem of optimal recovery of the operator Q by inaccurate values of the first N components $x_1, \dots, x_N$.

A series problems of optimal recovery of derivatives [1] and solutions of partial differential equations [4], [5] are reduced to the problems of such kind. In the cited papers the sequence $\{\mu_j\}_{j\in\mathbb{N}}$ was monotonic which made significantly easy to obtain an optimal recovery method. In this paper we consider the case when the sequence $\{\mu_j\}_{j\in\mathbb{N}}$ has less restrictive conditions.

We now proceed to the accurate statement of the problem. Put

$$W = \{\, x \in X : \|x\|_X \le 1 \,\}.$$

We assume that for every $x \in W$ we know a vector $y = (y_1, \dots, y_N)$ such that

$$\|I_N x - y\|_{l_2^N} = \left(\sum_{j=1}^{N} |x_j - y_j|^2 \right)^{1/2} \le \delta$$

(here $I_N x = (x_1, \dots, x_N)$). Any map $\varphi\colon l_2^N \to l_2$ is admitted as a recovery method. For a given method φ the error of recovery is defined by the equlity

$$e(Q, W, I_N, \delta, \varphi) = \sup_{\substack{x \in W,\ y \in l_2^N \\ \|I_N x - y\|_{l_2^N} \le \delta}} \|Qx - \varphi(y)\|_{l_2}.$$

The quantity

$$E(Q, W, I_N, \delta) = \inf_{\varphi\colon l_2^N \to l_2} e(Q, W, I_N, \delta, \varphi)$$

is called the error of optimal recovery, and a method delivering the lower bound is called optimal recovery method of the operator Q on the class W by the information I_N given with the error in the l_2^N norm.

2. Main results

Assume that $\nu_1 < \ldots < \nu_N$, $\nu_{N+1} < \nu_{N+2} < \ldots$, and $\lim_{j\to+\infty} \mu_j/\nu_j = 0$. Denote by e_j, $j = 1, 2, \dots$, the standard basis in l_2

$$(e_j)_k = \begin{cases} 1, & k = j, \\ 0, & k \ne j, \end{cases} \qquad k = 1, 2, \dots .$$

We introduce the following notation

$$A = \max_{1 \le j \le N} \frac{\mu_j}{\nu_j}, \qquad B = \max_{j > N} \frac{\mu_j}{\nu_j}.$$

Let $1 \le p \le N$, $q > N$, and $p \le r \le N$ be such that

$$\frac{\mu_p}{\nu_p} = A, \quad \frac{\mu_q}{\nu_q} = B, \quad \mu_r - B\nu_r = \max_{p \le j \le N} (\mu_j - B\nu_j)$$

(for uniqueness we assume that p is maximum and q with r are minimum of such numbers). Moreover, let s_{k+1} be the maximal number such that $s_k < s_{k+1} \le r$ and

$$\frac{\mu_{s_{k+1}} - \mu_{s_k}}{\nu_{s_{k+1}} - \nu_{s_k}} = \max_{s_k < j \le r} \frac{\mu_j - \mu_{s_k}}{\nu_j - \nu_{s_k}}, \quad k = 0, 1, \dots, m-1,$$

where $s_0 = p$, $s_m = r$. Set

$$J_k = \left\{ j \in \mathbb{N} \cap [1, N] : \frac{\mu_j}{\nu_j} > \frac{\mu_{s_{k+1}} - \mu_{s_k}}{\nu_{s_{k+1}} - \nu_{s_k}} \right\}, \quad k = 0, \dots, m-1,$$

$$J_m = \left\{ j \in \mathbb{N} \cap [1, N] : \frac{\mu_j}{\nu_j} > B \right\}.$$

If we plot the points (ν_j, μ_j) on the plane then the geometrical meaning of the introduced quantities can be seen from Fig. 1, where $m = 3$ and D_1 is the region where points from J_1 are lying.

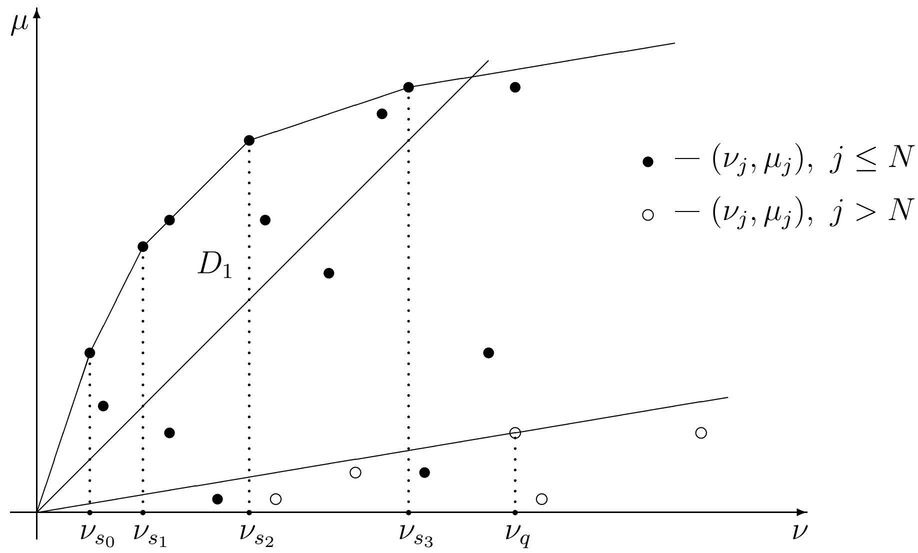

FIGURE 1.

Theorem 1. *If $B \ge A$ then for all $\delta > 0$*

$$E(Q, W, I_N, \delta) = \sqrt{\frac{\mu_q}{\nu_q}},$$

and the method $\widehat{\varphi}(y) = 0$ is optimal. If $B < A$ then (*i*) *for* $\delta \geq \dfrac{1}{\sqrt{\nu_p}}$

$$E(Q,W,I_N,\delta)=\sqrt{\frac{\mu_p}{\nu_p}},$$

and the method $\widehat{\varphi}(y)=0$ *is optimal;*

(*ii*) *for* $\dfrac{1}{\sqrt{\nu_{s_{k+1}}}} \leq \delta < \dfrac{1}{\sqrt{\nu_{s_k}}}$, $k=0,1,\ldots,m-1$,

$$E(Q,W,I_N,\delta)=\sqrt{\mu_{s_k}\frac{\nu_{s_{k+1}}\delta^2-1}{\nu_{s_{k+1}}-\nu_{s_k}}+\mu_{s_{k+1}}\frac{1-\delta^2\nu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}},$$

and the method

$$\widehat{\varphi}(y)=\sum_{j\in J_k}\eta_j\left(1+\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\mu_{s_k}\nu_{s_{k+1}}-\mu_{s_{k+1}}\nu_{s_k}}\nu_j\right)^{-1}y_je_j$$

is optimal;

(*iii*) *for* $\delta < \dfrac{1}{\sqrt{\nu_r}}$

$$E(Q,W,I_N,\delta)=\sqrt{\mu_r\delta^2+\mu_q\frac{1-\delta^2\nu_r}{\nu_q}},$$

and the method

$$\widehat{\varphi}(y)=\sum_{j\in J_m}\eta_j\left(1+\frac{\mu_q}{\mu_r\nu_q-\mu_q\nu_r}\nu_j\right)^{-1}y_je_j$$

is optimal.

It should be noted that for all $\delta>0$ except the points $1/\sqrt{\nu_{s_k}}$, $k=0,1,\ldots,m$, the optimal recovery method does not change for sufficiently small variation of δ, thus it is stable with respect to the error of input data.

We return to the problem of optimal recovery of the wave equation solution. If $f(\cdot)\in W_2^n([0,\pi])$ then

$$f(x)=\sum_{j=1}^{\infty}a_j(f)\sin jx,$$

where

$$\sum_{j=1}^{\infty}j^{2n}a_j^2(f)\leq 1,$$

that is $\nu_j = j^{2n}$. It follows from (2) that $\mu_j = \cos^2 jT$. In accordance with our notation

$$A = \max_{1\le j\le N} \frac{\cos^2 jT}{j^{2n}} = \frac{\cos^2 pT}{p^{2n}}, \quad B = \max_{j>N} \frac{\cos^2 jT}{j^{2n}} = \frac{\cos^2 qT}{q^{2n}},$$

r is defined from the condition

$$\cos^2 rT - Br^{2n} = \max_{p\le j\le N} (\cos^2 jT - Bj^{2n}),$$

the sequence s_{k+1} is defined by equalities

$$\frac{\cos^2 s_{k+1}T - \cos^2 s_kT}{s_{k+1}^{2n} - s_k^{2n}} = \max_{s_k<j\le r} \frac{\cos^2 jT - \cos^2 s_kT}{j^{2n} - s_k^{2n}}, \quad k = 0, 1, \dots, m-1,$$

where $s_0 = p$, $s_m = r$, and

$$J_k = \left\{ j \in \mathbb{N} \cap [1, N] : \frac{\cos^2 jT}{j^{2n}} > \frac{\cos^2 s_{k+1}T - \cos^2 s_kT}{s_{k+1}^{2n} - s_k^{2n}} \right\}, \quad k = 0, \dots, m-1,$$

$$J_m = \left\{ j \in \mathbb{N} \cap [1, N] : \frac{\cos^2 jT}{j^{2n}} > B \right\}.$$

It follows from Theorem 1

Corollary 1. *If $B \ge A$ for all $\delta > 0$ the method*

$$u(x, T) \approx 0$$

is an optimal recovery method for the wave equation and for its error the equality

$$E(T, W_2^n([0, \pi]), F_\delta^N) = \frac{|\cos qT|}{q^n}$$

holds. If $B < A$ then

(i) for $\delta \ge p^{-n}$

$$E(T, W_2^n([0, \pi]), F_\delta^N) = \frac{|\cos pT|}{p^n}$$

and the method $u(x, T) \approx 0$ is optimal;

(ii) for $s_{k+1}^{-n} \le \delta < s_k^{-n}$, $k = 0, 1, \dots, m-1$,

$$E(T, W_2^n([0, \pi]), F_\delta^N) = \sqrt{\frac{s_{k+1}^{2n}\delta^2 - 1}{s_{k+1}^{2n} - s_k^{2n}} \cos^2 s_kT + \frac{1 - \delta^2 s_k^{2n}}{s_{k+1}^{2n} - s_k^{2n}} \cos^2 s_{k+1}T}$$

and the method

$$u(x,T) \approx \sum_{j\in J_k}\left(1+\frac{\cos^2 s_{k+1}T-\cos^2 s_kT}{s_{k+1}^{2n}\cos^2 s_kT-s_k^{2n}\cos^2 s_{k+1}T}j^{2n}\right)^{-1} y_j\cos jT\sin jx$$

is optimal;

(*iii*) *for* $\delta < r^{-n}$

$$E(T,W_2^n([0,\pi]),F_\delta^N)=\sqrt{\delta^2\cos^2 rT+\frac{1-\delta^2r^{2n}}{q^{2n}}\cos^2 qT}$$

and the method

$$u(x,T)\approx\sum_{j\in J_m}\left(1+\frac{\cos^2 qT}{q^{2n}\cos^2 rT-r^{2n}\cos^2 qT}j^{2n}\right)^{-1}y_j\cos jT\sin jx$$

is optimal.

3. Proofs

We begin with a preliminary result which describes properties of the sequences $\{\mu_{s_k}\}$ and $\{\nu_{s_k}\}$.

Lemma 1. *The sequences*

$$\left\{\frac{\mu_{s_k}}{\nu_{s_k}}\right\} \quad \text{and} \quad \left\{\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}\right\}$$

are strictly monotone decreasing and for all $1\le j<s_k$

$$\frac{\mu_{s_k}-\mu_j}{\nu_{s_k}-\nu_j}\ge\frac{\mu_{s_k}-\mu_{s_{k-1}}}{\nu_{s_k}-\nu_{s_{k-1}}}. \tag{4}$$

Proof. We prove that the sequence $\{\mu_{s_k}/\nu_{s_k}\}$ is strictly monotone decreasing. It follows from the definition of p that for all $i\ge1$

$$\frac{\mu_p}{\nu_p}=\frac{\mu_{s_0}}{\nu_{s_0}}>\frac{\mu_{s_i}}{\nu_{s_i}}.$$

Assuming that for all $i\ge k$

$$\frac{\mu_{s_{k-1}}}{\nu_{s_{k-1}}}>\frac{\mu_{s_i}}{\nu_{s_i}}, \tag{5}$$

we prove that for all $i\ge k+1$

$$\frac{\mu_{s_k}}{\nu_{s_k}}>\frac{\mu_{s_i}}{\nu_{s_i}}. \tag{6}$$

It follows from the definition of s_k that for all $i \ge k+1$

$$\frac{\mu_{s_k} - \mu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}} > \frac{\mu_{s_i} - \mu_{s_{k-1}}}{\nu_{s_i} - \nu_{s_{k-1}}}.$$

Hence,

$$\mu_{s_k}\nu_{s_i} - \mu_{s_k}\nu_{s_{k-1}} - \mu_{s_{k-1}}\nu_{s_i} > \mu_{s_i}\nu_{s_k} - \mu_{s_{k-1}}\nu_{s_k} - \mu_{s_i}\nu_{s_{k-1}}.$$

Consequently,

$$\begin{aligned}\nu_{s_k}\nu_{s_i}\left(\frac{\mu_{s_k}}{\nu_{s_k}} - \frac{\mu_{s_i}}{\nu_{s_i}}\right) > \nu_{s_i}\nu_{s_{k-1}}&\left(\frac{\mu_{s_{k-1}}}{\nu_{s_{k-1}}} - \frac{\mu_{s_i}}{\nu_{s_i}}\right)\\ &- \nu_{s_{k-1}}\nu_{s_k}\left(\frac{\mu_{s_{k-1}}}{\nu_{s_{k-1}}} - \frac{\mu_{s_k}}{\nu_{s_k}}\right).\end{aligned}$$

In view of (5) and the fact that $\nu_{s_i} > \nu_{s_k}$ we have

$$\begin{aligned}\nu_{s_k}\nu_{s_i}\left(\frac{\mu_{s_k}}{\nu_{s_k}} - \frac{\mu_{s_i}}{\nu_{s_i}}\right) &> \nu_{s_k}\nu_{s_{k-1}}\left(\frac{\mu_{s_{k-1}}}{\nu_{s_{k-1}}} - \frac{\mu_{s_i}}{\nu_{s_i}} - \frac{\mu_{s_{k-1}}}{\nu_{s_{k-1}}} + \frac{\mu_{s_k}}{\nu_{s_k}}\right)\\ &= \nu_{s_k}\nu_{s_{k-1}}\left(\frac{\mu_{s_k}}{\nu_{s_k}} - \frac{\mu_{s_i}}{\nu_{s_i}}\right).\end{aligned}$$

Consequently, (6) holds.

It follows from the choice of the sequence s_k that

$$\frac{\mu_{s_{k+1}} - \mu_{s_{k-1}}}{\nu_{s_{k+1}} - \nu_{s_{k-1}}} < \frac{\mu_{s_k} - \mu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}}.$$

Hence

$$\mu_{s_{k+1}} - \mu_{s_{k-1}} < (\mu_{s_k} - \mu_{s_{k-1}})\frac{\nu_{s_{k+1}} - \nu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}}.$$

Then

$$\begin{aligned}\mu_{s_{k+1}} - \mu_{s_k} &= (\mu_{s_{k+1}} - \mu_{s_{k-1}}) - (\mu_{s_k} - \mu_{s_{k-1}})\\ &< (\mu_{s_k} - \mu_{s_{k-1}})\left(\frac{\nu_{s_{k+1}} - \nu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}} - 1\right) = (\mu_{s_k} - \mu_{s_{k-1}})\frac{\nu_{s_{k+1}} - \nu_{s_k}}{\nu_{s_k} - \nu_{s_{k-1}}}.\end{aligned}$$

Consequently,

$$\frac{\mu_{s_{k+1}} - \mu_{s_k}}{\nu_{s_{k+1}} - \nu_{s_k}} < \frac{\mu_{s_k} - \mu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}}.$$

To prove (4), first, we show that for $s_{k-1} \le j < s_k$

$$\frac{\mu_{s_k} - \mu_j}{\nu_{s_k} - \nu_j} \ge \frac{\mu_{s_k} - \mu_{s_{k-1}}}{\nu_{s_k} - \nu_{s_{k-1}}}. \tag{7}$$

It follows from the definition of s_k that

$$\frac{\mu_j-\mu_{s_{k-1}}}{\nu_j-\nu_{s_{k-1}}}\le\frac{\mu_{s_k}-\mu_{s_{k-1}}}{\nu_{s_k}-\nu_{s_{k-1}}}.$$

Cosequently,

$$\mu_j\le\mu_{s_{k-1}}+(\mu_{s_k}-\mu_{s_{k-1}})\frac{\nu_j-\nu_{s_{k-1}}}{\nu_{s_k}-\nu_{s_{k-1}}}.$$

Thus,

$$\begin{aligned}\mu_{s_k}-\mu_j\ge\mu_{s_k}-\mu_{s_{k-1}}-(\mu_{s_k}-\mu_{s_{k-1}})\frac{\nu_j-\nu_{s_{k-1}}}{\nu_{s_k}-\nu_{s_{k-1}}}\\ =(\mu_{s_k}-\mu_{s_{k-1}})\frac{\nu_{s_k}-\nu_j}{\nu_{s_k}-\nu_{s_{k-1}}}.\end{aligned}$$

Hence (7) holds. Let $j<s_{k-1}$. First, we note that the sequence $\{\mu_{s_k}\}$ is monotone increasing. Indeed, if for any k, $\mu_{s_{k-1}}>\mu_{s_k}$, then

$$\frac{\mu_{s_k}-\mu_{s_{k-1}}}{\nu_{s_k}-\nu_{s_{k-1}}}<0,$$

and in view of monotone decreasing we have

$$\frac{\mu_{s_m}-\mu_{s_{m-1}}}{\nu_{s_m}-\nu_{s_{m-1}}}<0,$$

that is $\mu_{s_{m-1}}>\mu_{s_m}=\mu_r$. Then

$$\mu_{s_{m-1}}-B\nu_{s_{m-1}}\ge\mu_{s_{m-1}}-B\nu_r>\mu_r-B\nu_r,$$

which contradicts the definition of r. We show now that for all $i<s_k$, $\mu_i\le\mu_{s_k}$. Let $l\le k$ and $s_{l-1}<i<s_l$. Then $\mu_i\le\mu_{s_l}$, since otherwise we have

$$\frac{\mu_i-\mu_{s_{l-1}}}{\nu_i-\nu_{s_{l-1}}}>\frac{\mu_{s_l}-\mu_{s_{l-1}}}{\nu_i-\nu_{s_{l-1}}}\ge\frac{\mu_{s_l}-\mu_{s_{l-1}}}{\nu_{s_l}-\nu_{s_{l-1}}},$$

which contradicts the definition of s_l. Thus, $\mu_i\le\mu_{s_l}\le\mu_{s_k}$. In view of $j<s_{k-1}$, by the fact proved above it follows that $\mu_j\le\mu_{s_{k-1}}$. We have

$$\mu_{s_k}-\mu_{s_{k-1}}\le\mu_{s_k}-\mu_j\le(\mu_{s_k}-\mu_j)\frac{\nu_{s_k}-\nu_{s_{k-1}}}{\nu_{s_k}-\nu_j},$$

which yields (7). □

Proof of Theorem 1. Consider the extremal problem

$$\sum_{j=1}^{\infty}\mu_j|x_j|^2\to\max,\quad\sum_{j=1}^{N}|x_j|^2\le\delta^2,\quad\sum_{j=1}^{\infty}\nu_j|x_j|^2\le1.$$

Put $u_j=|x_j|^2$, $j\in\mathbb{N}$, and rewrite this problem in the following form:

$$\sum_{j=1}^{\infty}\mu_ju_j\to\max,\quad \sum_{j=1}^{N}u_j\le\delta^2,\quad \sum_{j=1}^{\infty}\nu_ju_j\le 1,\quad u_j\ge 0. \tag{8}$$

We define the Lagrange function for this problem

$$\mathcal{L}(u,\lambda_1,\lambda_2)=\sum_{j=1}^{N}(-\mu_j\lambda_1+\lambda_2\nu_j)u_j+\sum_{j=N+1}^{\infty}(-\mu_j+\lambda_2\nu_j)u_j,$$

where $u=\{u_j\}_{j\in\mathbb{N}}$, and λ_1,λ_2 are the Lagrange multipliers.

It follows from [2] (see also [3]) that if there exist such $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ that for the sequence $\widehat{u}=\{\widehat{u}_j\}_{j\in\mathbb{N}}$ admissible in (8) the conditions

$$(a)\quad \min_{u_j\ge 0}\mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2)=\mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2),$$

$$(b)\quad \widehat{\lambda}_1\left(\sum_{j=1}^{N}\widehat{u}_j-\delta^2\right)=0,\quad \widehat{\lambda}_2\left(\sum_{j=1}^{\infty}\nu_j\widehat{u}_j-1\right)=0,$$

hold, then $\widehat{u}$ is the solution of the problem (8) and its value is equal to $\widehat{\lambda}_1\delta^2+\widehat{\lambda}_2$. Moreover, if for all $y\in l_2^N$ there exists a solution x_y of the extremal problem

$$\widehat{\lambda}_1\|I_Nx-y\|^2_{l_2^N}+\widehat{\lambda}_2\|x\|^2_X\to\min,\quad x\in X, \tag{9}$$

then

$$E(Q,W,I_N,\delta)=\sqrt{\widehat{\lambda}_1\delta^2+\widehat{\lambda}_2}, \tag{10}$$

and the method

$$\widehat{\varphi}(y)=Qx_y$$

is optimal.

Problem (9) can be written in the form

$$\sum_{j=1}^{N}\left(\widehat{\lambda}_1(x_j-y_j)^2+\widehat{\lambda}_2\nu_jx_j^2\right)+\widehat{\lambda}_2\sum_{j=N+1}^{\infty}\nu_jx_j^2\to\min,\quad x\in X.$$

It is easy to verify that for fixed $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ its solution is

$$x_y=\sum_{j=1}^{N}\frac{\widehat{\lambda}_1}{\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j}y_je_j.$$

Therefore it is sufficient to find $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ and a sequence $\widehat{u}=\{\widehat{u}_j\}_{j\in\mathbb{N}}$ admissible in (8) for which conditions (a) and (b) hold. In this case the method

$$\widehat{\varphi}(y)=\sum_{j=1}^{N}\eta_j\frac{\widehat{\lambda}_1}{\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j}y_je_j \tag{11}$$

is optimal.

Let $B\geq A$. Put $\widehat{\lambda}_1=0$,

$$\widehat{\lambda}_2=\frac{\mu_q}{\nu_q},\quad \widehat{u}_q=\frac{1}{\nu_q},\quad \widehat{u}_j=0,\ j\neq q.$$

It is easy to verify that the sequence $\{\widehat{u}_j\}$ is admissible and conditions (b) hold. We have

$$\mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2)=\sum_{j=1}^{\infty}\left(-\mu_j+\frac{\mu_q}{\nu_q}\nu_j\right)u_j=\sum_{j=1}^{\infty}\nu_j\left(\frac{\mu_q}{\nu_q}-\frac{\mu_j}{\nu_j}\right)u_j\geq 0,$$

since

$$B=\frac{\mu_q}{\nu_q}\geq\max_{j\in N}\frac{\mu_j}{\nu_j}=A.$$

In view of the fact that $\mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2)=0$, condition (a) holds.

Let $B<A$. We start with the case (i). Put $\widehat{\lambda}_1=0$,

$$\widehat{\lambda}_2=\frac{\mu_p}{\nu_p},\quad \widehat{u}_p=\frac{1}{\nu_p},\quad \widehat{u}_j=0,\ j\neq p.$$

In this situation it is also easy to verify that the sequence $\{\widehat{u}_j\}$ is admissible and conditions (b) hold. Since

$$\frac{\mu_p}{\nu_p}=\max_{j\in N}\frac{\mu_j}{\nu_j}=A>B=\max_{j>N}\frac{\mu_j}{\nu_j},$$

in this case

$$\mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2)=\sum_{j=1}^{\infty}\left(-\mu_j+\frac{\mu_p}{\nu_p}\nu_j\right)u_j=\sum_{j=1}^{\infty}\nu_j\left(\frac{\mu_p}{\nu_p}-\frac{\mu_j}{\nu_j}\right)u_j\geq 0.$$

Since $\mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2)=0$, condition (a) holds.

We proceed to the case (ii). Let

$$\frac{1}{\sqrt{\nu_{s_{k+1}}}}\leq\delta<\frac{1}{\sqrt{\nu_{s_k}}}. \tag{12}$$

Put $\widehat{u}_j=0$ if $j\neq s_k,s_{k+1}$, and $\widehat{u}_{s_k}$ with $\widehat{u}_{s_{k+1}}$ we find from the condition

$$\widehat{u}_{s_k}+\widehat{u}_{s_{k+1}}=\delta^2, \tag{13}$$

$$\nu_{s_k}\widehat{u}_{s_k}+\nu_{s_{k+1}}\widehat{u}_{s_{k+1}}=1. \tag{14}$$

Then
$$\widehat{u}_{s_k}=\frac{\nu_{s_{k+1}}\delta^2-1}{\nu_{s_{k+1}}-\nu_{s_k}},\quad \widehat{u}_{s_{k+1}}=\frac{1-\nu_{s_k}\delta^2}{\nu_{s_{k+1}}-\nu_{s_k}}.$$
In view of (12) and (13) the sequence $\{\widehat{u}_j\}$ is admissible in (8). Put
$$\widehat{\lambda}_1=\mu_{s_k}-\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}\nu_{s_k},\quad \widehat{\lambda}_2=\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}.$$
First, we show that $\widehat{\lambda}_1,\widehat{\lambda}_2>0$. Since by Lemma 1 the sequence $\{\mu_{s_k}/\nu_{s_k}\}$ is monotone decreasing, we have
$$\widehat{\lambda}_1=\frac{\mu_{s_k}\nu_{s_{k+1}}-\mu_{s_{k+1}}\nu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}=\frac{\nu_{s_k}\nu_{s_{k+1}}}{\nu_{s_{k+1}}-\nu_{s_k}}\left(\frac{\mu_{s_k}}{\nu_{s_k}}-\frac{\mu_{s_{k+1}}}{\nu_{s_{k+1}}}\right)>0.$$
It follows from the definition of r that $\mu_r-B\nu_r>\mu_{s_{m-1}}-B\nu_{s_{m-1}}$. Thus, since $s_m=r$,
$$\frac{\mu_{s_m}-\mu_{s_{m-1}}}{\nu_{s_m}-\nu_{s_{m-1}}}>B.$$
It follows from monotone decreasing of the sequence
$$\left\{\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}\right\}$$
that
$$\widehat{\lambda}_2>B\ge 0. \tag{15}$$

It follows from (13) that condition (b) holds. We prove that condition (a) also holds. We show that for all $u_j\ge 0$
$$\mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2)\ge 0. \tag{16}$$

If $j>N$, then taking into account (15),
$$-\mu_j+\widehat{\lambda}_2\nu_j=\nu_j\left(\widehat{\lambda}_2-\frac{\mu_j}{\nu_j}\right)>\nu_j\left(B-\frac{\mu_j}{\nu_j}\right)\ge 0.$$
If $s_k\le j\le N$, then in view of the definition of s_k
$$-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j=(\nu_j-\nu_{s_k})\left(\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}-\frac{\mu_j-\mu_{s_k}}{\nu_j-\nu_{s_k}}\right)\ge 0.$$
For $1\le j<s_k$, taking into account (4), we have
$$-\mu_j+\widehat{\lambda}_1+\widehat{\lambda}_2\nu_j=(\nu_{s_k}-\nu_j)\left(\frac{\mu_{s_k}-\mu_j}{\nu_{s_k}-\nu_j}-\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}\right)\ge 0.$$
Hence the inequality (16) is proved, and since $\mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2)=0$, it is proved that condition (a) holds. Thus, substituting $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ in (10)

and (11), we obtain the error of optimal recovery and optimality of the method

$$\varphi(y)=\sum_{j=1}^{N}\eta_j\left(1+\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\mu_{s_k}\nu_{s_{k+1}}-\mu_{s_{k+1}}\nu_{s_k}}\nu_j\right)^{-1}y_je_j.$$

We show that the method in which the summation is taken over not all $1\le j\le N$ but only over the set J_k is also optimal. Let

$$J_k=\{i_1,\dots,i_{\widetilde N}\}.$$

Consider the same optimal recovery problem but with the information operator

$$I_{J_k}x=(x_{i_1},\dots,x_{i_{\widetilde N}}).$$

It follows from Lemma 1 that for all $j=0,1,\dots,k$

$$\frac{\mu_{s_j}}{\nu_{s_j}}>\frac{\mu_{s_j}-\mu_{s_{j-1}}}{\nu_{s_j}-\nu_{s_{j-1}}}>\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\nu_{s_{k+1}}-\nu_{s_k}}.$$

Therefore for the new information operator I_{J_k} the sequence s_j, $j=0,1,\dots,\widetilde m$, $\widetilde m\ge k$, will not change. Further, the following two cases may occur: $\widetilde m>k$ and $\widetilde m=k$. Consider the first case (the second one will follow by similar assertions from the case (iii)). By proved above for the case when

$$\frac{1}{\sqrt{\nu_{s_{k+1}}}}\le\delta<\frac{1}{\sqrt{\nu_{s_k}}},$$

the error of optimal recovery depends only on two points (μ_{s_k},ν_{s_k}) and $(\mu_{s_{k+1}},\nu_{s_{k+1}})$. Therefore,

$$E(Q,W,I_{J_k},\delta)=E(Q,W,I_N,\delta),$$

and the method

$$\widehat\varphi(\widetilde y)=\sum_{j=1}^{\widetilde N}\eta_{i_j}\left(1+\frac{\mu_{s_{k+1}}-\mu_{s_k}}{\mu_{s_k}\nu_{s_{k+1}}-\mu_{s_{k+1}}\nu_{s_k}}\nu_{i_j}\right)^{-1}y_{i_j}e_{i_j},$$

$$\widetilde y=(y_{i_1},\dots,y_{i_{\widetilde N}}),$$

is optimal. We estimate this method for the information operator I_N. Let $x\in W$, $y\in l_2^N$, and $\|I_Nx-y\|_{l_2^N}\le\delta$. Then $\|I_{J_k}x-\widetilde y\|_{l_2^{\widetilde N}}\le\delta$. Thus,

$$\|Q-\widehat\varphi(y)\|_{l_2}=\|Q-\widehat\varphi(\widetilde y)\|_{l_2}\le E(Q,W,I_{J_k},\delta)=E(Q,W,I_N,\delta).$$

It means that the method $\widehat\varphi$ is optimal for the information operator I_N.

For the case (iii) we put

$$\widehat{\lambda}_1=\mu_r-\frac{\mu_q}{\nu_q}\nu_r,\quad \widehat{\lambda}_2=\frac{\mu_q}{\nu_q},\quad \widehat{u}_r=\delta^2,\quad \widehat{u}_q=\frac{1-\delta^2\nu_r}{\nu_q},$$
$$\widehat{u}_j=0,\ j\neq r,q.$$

It follows from the definition r that

$$\widehat{\lambda}_1>\mu_p-B\nu_p=\nu_p(A-B)>0.$$

It is easy to verify that the sequence $\{\widehat{u}_j\}$ is admissible and condition (b) holds. Since in this case again $\mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2)=0$, to prove the realization of condition (a) it remains to prove that for all $u_j\geq 0$ the inequality (16) holds. In the present case the Lagrange function has the following form

$$\mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2)=\sum_{j=1}^N\left(-\mu_j+\mu_r-\frac{\mu_q}{\nu_q}\nu_r+\frac{\mu_q}{\nu_q}\nu_j\right)u_j$$
$$+\sum_{j=N+1}^\infty\left(-\mu_j+\frac{\mu_q}{\nu_q}\nu_j\right)u_j=\sum_{j=1}^N\left(\left(\mu_r-\frac{\mu_q}{\nu_q}\nu_r\right)-\left(\mu_j-\frac{\mu_q}{\nu_q}\nu_j\right)\right)u_j$$
$$+\sum_{j=N+1}^\infty\nu_j\left(\frac{\mu_q}{\nu_q}-\frac{\mu_j}{\nu_j}\right)u_j.$$

Every term in the first sum is nonnegative in the view of the definition of r, and every term of the second sum is nonnegative in the view of the definition of q. Substituting $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ in (10) and (11), we obtain the error of optimal recovery and optimality of the method

$$\varphi(y)=\sum_{j=1}^N\eta_j\left(1+\frac{\mu_q}{\mu_r\nu_q-\mu_q\nu_r}\nu_j\right)^{-1}y_je_j. \tag{17}$$

The arguments similar to those which where used in the proof of the case (ii) show that in the method (17) the points $(\nu_j,\mu_j)\notin J_m$ may be discarded. In this case the obtained method will be also optimal but the number of input data which are used in general will be reduced. □

References

[1] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error", *Mat. Sb.* **193** (2002), 79–100; English transl. in *Sb. Math.* **193** (2002).

[2] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from inaccurate information about a spectrum and inequalities

for derivatives", *Funkc. analiz i ego prilozh.* **37** (2003), 51–64; English transl. in *Funct. Anal and Its Appl.*, **37** (2003).

[3] K. Yu. Osipenko, "The Hardy–Littlewood–Pólya inequality for analytic functions from Hardy–Sobolev spaces", *Mat. Sb.* **197** (2006), 15–34; English transl. in *Sbornic: Mathematics* **197** (2006), 315–334.

[4] G. G. Magaril-Il'yaev, K. Yu. Osipenko, and V. M. Tikhomirov, "On optimal recovery of heat equation solutions". In: Approximation Theory: A volume dedicated to B. Bojanov (D. K. Dimitrov, G. Nikolov, and R. Uluchev, Eds.), 163–175, Sofia: Marin Drinov Academic Publishing House, 2004.

[5] K. Yu. Osipenko, "On recovery of the Dirichlet problem solution by inaccurate input data", *Vladikavkaz Mat. Zh.*, **6**, 4 (2004), 55–62.

"MATI" — Russian State Technological University